Kinetic equation for a soliton gas, its hydrodynamic reductions and symmetries

G.A. El[1], A.M. Kamchatnov[2], M.V. Pavlov[3] and S.A. Zykov[4]

[1] Department of Mathematical Sciences, Loughborough University, UK
[2]Institute of Spectroscopy, Russian Academy of Sciences, Troitsk, Moscow Region, Russia
[3] Lebedev Physical Institute, Russian Academy of Sciences, Moscow
[4] SISSA, Trieste, Italy, and
Institute of Metal Physics, Urals Division of Russian Academy of Sciences, Ekaterinburg, Russia

Abstract

We study a new class of kinetic equations describing nonequilibrium macroscopic dynamics of soliton gases with elastic collisions. These equations represent nonlinear integro-differential systems and have a novel structure, which we investigate by studying in detail the class of N-component 'cold-gas' hydrodynamic reductions. We prove that these reductions represent integrable linearly degenerate hydrodynamic type systems for arbitrary N which is a strong indication to integrability of the full kinetic equation. We derive explicit representations for the Riemann invariants and characteristic velocities of the hydrodynamic reductions in terms of the soliton gas component densities and construct a number of exact solutions having special properties (quasi-periodic, self-similar). Hydrodynamic symmetries are then derived and investigated.

1 Introduction

In the recent paper [38] V.E. Zakharov has put forward a programme for the construction of the theory of wave turbulence in integrable systems. Such an unconventional union of integrability and stochasticity has a clear physical motivation: nonlinear dispersive waves, while often being successfully modeled by integrable systems, could demonstrate very complex behaviour calling for a statistical description characteristic of the classical turbulence theories.

One of the important problems arising in this connection is the description of "soliton gases" — random distributions of solitons which can be mathematically defined in terms of generalized reflectionless potentials with shift invariant probability measure on them (see e.g. [24]). Due to isospectrality of the "primitive" microscopic evolution, the macroscopic dynamics of a homogeneous soliton gas is trivial (for strongly integrable systems, such as the Korteweg – de Vries (KdV), nonlinear Schrödinger (NLS) or Kadomtsev-Petviashvili (KP-II) equations — see [38]), namely, all the statistical characteristics can be specified arbitrarily at the initial moment and remain unchanged in time. However, if the soliton gas is spatially inhomogeneous, i.e. if the probability distribution function depends on the space coordinate, then nontrivial macroscopic dynamics occurs due to phase shifts of individual solitons in their collisions with each other. The approximate kinetic equation describing spatial evolution of the soliton distribution function in a rarefied gas of the KdV solitons, when these phase shifts can be taken into account explicitly, was derived by Zakharov back in 1971 [35].

Generalization of Zakharov's kinetic equation to the case of soliton gas of finite density has been made possible rather recently [5] and required consideration of the *thermodynamic limit* of the Whitham modulation equations associated with finite-gap potentials (note that the idea to use the Whitham equations for the modeling of a turbulent motion – a "deterministic analoque of turbulence" – was first proposed by P.D. Lax [25]). In the thermodynamic limit, the nonlinear interacting wave modes transform into randomly distributed localised states (solitons) and the modulation system assumes the form of a nonlinear kinetic equation. This new kinetic equation was extended, using physical reasoning, in [7] to other integrable systems with two-particle elastic interactions of solitons (i.e. when multi-particle effects are absent).

The kinetic equation for solitons in general form represents a nonlinear integro-differential system

$$\begin{aligned} & f_t + (sf)_x = 0\,, \\ & s(\eta) = S(\eta) + \frac{1}{\eta}\int_0^\infty G(\eta,\mu) f(\mu)[s(\mu)-s(\eta)]d\mu\,. \end{aligned} \tag{1}$$

Here $f(\eta) \equiv f(\eta,x,t)$ is the distribution function and $s(\eta) \equiv s(\eta,x,t)$ is the associated transport velocity. The (given) functions $S(\eta)$ and $G(\eta,\mu)$ do not depend on x and t. The function $G(\eta,\mu)$ is assumed to be symmetric, i.e. $G(\eta,\mu) = G(\mu,\eta)$. The choice

$$S(\eta) = 4\eta^2\,, \qquad G(\eta,\mu) = \log\left|\frac{\eta-\mu}{\eta+\mu}\right| \tag{2}$$

corresponds to the KdV case, $\varphi_t - 6\varphi\varphi_x + \varphi_{xxx} = 0$ (see [5]). In the KdV context, $\eta \geq 0$ is a real-valued spectral parameter (to be precise, before the passage to the continuum limit one has $\lambda_k = -\eta_k^2$, where λ_k, $k = 1,\ldots,N$ are the discrete eigenvalues of the Schrödinger operator), thus the function $f(\eta,x,t)$ is the distribution function of solitons over spectrum so that $\kappa = \int_0^\infty f(\eta)d\eta = \mathcal{O}(1)$ is the spatial density of solitons. If $\kappa \ll 1$, the first order approximation of (1), (2) yields Zakharov's kinetic equation for a dilute gas of KdV solitons [35].

The quantity $S(\eta)$ in (1) has a natural meaning of the velocity of an isolated (free) soliton with the spectral parameter η and the function $\frac{1}{\eta}G(\eta,\mu)$ is the expression for a phase shift of this soliton occurring after its collision with another soliton having the spectral parameter $\mu < \eta$. Then $s(\eta,x,t)$ acquires the meaning of the self-consistently defined mean local velocity of solitons with the spectral parameter close to η (see [7]).

Theory of nonlocal kinetic equations of the form (1) is not developed yet. Possible approaches to their treatment were discussed in [2] in connection with special classes of exact solutions for the Boltzmann kinetic equation for Maxwellian particles. The derivation of (1), (2) as a certain (albeit singular) limit of the integrable KdV-Whitham system suggests that this new kinetic equation is also an integrable system, at least for special choices of functions $G(\eta, \mu)$. A natural question arising in this connection is: what is the exact meaning of integrability for the equations of the type (1)?

Integrability of kinetic equations has been the subject of intensive studies in recent decades. For instance, integrability of the collisionless Boltzmann equation (which is sometimes called the Vlasov equation) can be defined in terms of two other closely connected (even equivalent in some sense) objects: the Benney hydrodynamic chain [3], [36], [15] and the dispersionless limit of the Kadomtsev–Petviashvili equation ([22, 23]. It turns out that all these three different nonlinear partial differential equations possess the same infinite set of N-component hydrodynamic reductions parametrized by N arbitrary functions of a single variable [16, 17] (we note that the solutions to these N-component reductions are parametrized, in their turn, by another N arbitrary functions of a single variable). This property was used in Ferapontov & Khusnutdinova [10, 11](see also [37], [12], [18], [27]) when introducing the integrability criterion for a wide class of kinetic equations, corresponding hydrodynamic chains and 2+1 quasilinear equations. Moreover, it was proved in [29] that the existence of at least one N-component hydrodynamic reduction written in the so-called *symmetric* form is sufficient for integrability in the sense of [10]. Another possible approach to analyse an integrable kinetic equation is to use the fact that it possesses infinitely many particular solutions determined by the corresponding hydrodynamic reductions (see [27] for details).

The distinctive feature of the kinetic equation (1) is its nonlocal structure, which represents an obstacle to the direct application to it of the approaches developed in [29] and [27]. For instance, the possibility of an explicit construction of symmetric hydrodynamic reductions (and even the existence of such reductions) for (1) are open questions at the moment. In this paper, we study a particular, yet probably the most important from the viewpoint of capturing the essential properties of the full equation, family of the 'cold-gas' N-component hydrodynamic reduction of (1) obtained via the delta-functional ansatz for the distribution function $f(\eta, x, t) = \sum_{i=1}^{N} f^i(x,t)\delta(\eta - \eta_i)$. We prove that these reductions represent linearly degenerate semi-Hamiltonian (integrable) systems of hydrodynamic type (see [28] and [9]) and can be explicitly represented in the Riemann invariant form for arbitrary N. The corresponding characteristic velocities, conservation law densities and symmetries (commuting flows) are described in terms of the so-called Stäckel matrices determined by $N(N-1)$ linear functions of a single variable.

Integrability, for arbitrary N, of the class of the hydrodynamic reductions studied in this paper is a serious argument in favour of integrability of the full nonlocal kinetic equation (1), at least for certain choices of the functions $S(\lambda)$ and $G(\lambda, \mu)$ in the integral closure equation. Of course, such an outcome does not look surprising for the particular choice (2) of $S(\lambda)$ and $G(\lambda, \mu)$ corresponding to the thermodynamic limit of the integrable KdV-Whitham equations but our analysis suggests that the general integro-differential kinetic equation (1) is a representative of a whole new unexplored class of integrable equations with potentially important physical applications.

The structure of the paper is as follows. In Section 2 we outline the derivation of the kinetic equations for the gas of the KdV solitons following the thermodynamic limit procedure of [5] and extending it to the entire KdV-Whitham hierarchy. We then introduce the generalized equation (1), and in Section 3 consider its N-component 'cold-gas' hydrodynamic reductions having the form of hydrodynamic conservation laws. We then formulate our main Theorem 3.1 stating that the hydrodynamic reductions under study are linearly degenerate and integrable (in Tsarev's generalised hodograph sense) hydrodynamic type systems *for any* N. Section 4 is devoted to the account of the main results of the theory of linearly degenerate hydrodynamic type systems. In Section 5 we prove the statement of the main Theorem 3.1 for the case $N = 3$ by explicitly constructing the corresponding Stäckel matrix and presenting expressions for the Riemann invariants and characteristic velocities in terms of the conserving component densities. We also construct two distinguished families of exact solutions (self-similar and quasi-periodic) to the 3-component reduction. In Section 6, the existence of the Riemann invariant parametrization of the cold-gas hydrodynamic reduction, via a single Stäckel matrix, is proved for arbitrary N, which enables us to complete the proof of the main Theorem 3.1 for a general case. In Section 7, we derive explicit expressions for the Riemann invariants and characteristic velocities in terms of the component densities. And at last, in Section 8 we derive hydrodynamic symmetries (commuting flows) of the N-component hydrodynamic reductions under study and then extract the family of linearly degenerate commuting flows.

2 Kinetic equation for a soliton gas as the thermodynamic limit of the Whitham modulation system

We start with an outline of the derivation of the kinetic equation for the gas of the KdV solitons as the thermodynamic limit of the KdV-Whitham system following [5]. We then naturally extend this derivation to the whole Whitham-KdV hierarchy.

Let us consider the Whitham modulation system associated with the N-gap potentials $u_N(x,t)$ of the KdV equation. This is most conveniently represented as a single generation equation in the [13] form:

$$(dp_N)_t = (dq_N)_x \,, \tag{3}$$

where dp_N and dq_N are the quasimomentum and quasienergy differentials defined on the two-sheeted hyperelliptic Riemann surface of genus N :

$$\Gamma: \quad \mu^2(\lambda) = \prod_{j=1}^{2N+1} (\lambda - \lambda_j)\,, \qquad \lambda \in \mathbb{C}, \quad \lambda_j \in \mathbb{R}\,. \tag{4}$$

$$\lambda_1 < \lambda_2 < \cdots < \lambda_{2N} < \lambda_{2N+1}\,,$$

with cuts along spectral bands $[\lambda_1, \lambda_2]$, $\dots[\lambda_{2j-1}, \lambda_{2j}]$, $\dots$, $[\lambda_{2N+1}, \infty]$. We introduce the canonical system of cycles on Γ as follows (see Fig. 1): the α_j-cycle surrounds the j-th cut clockwise on the upper sheet, and the β_j- cycle is canonically conjugated to α_j's such that the closed contour β_j starts at λ_{2j} , goes to $+\infty$ on the upper sheet and returns to λ_{2j} on the lower sheet.

The meromorphic differentials dp_N and dq_N are uniquely defined by their asymptotic behaviour near $\lambda = -\infty$:

$$-\lambda \gg 1 \ : \qquad dp_N \sim -\frac{d\lambda}{(-\lambda)^{1/2}}\,, \qquad dq_N \sim (-\lambda)^{1/2} d\lambda \tag{5}$$

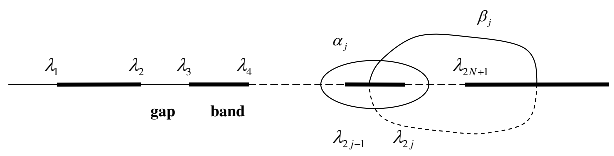

Figure 1: The canonical system of cycles on the hyperelliptic Riemann surface of genus N.

and the normalization

$$\oint_{\beta_i} dp_N = 0\,, \qquad \oint_{\beta_i} dq_N = 0\,, \qquad i = 1, \ldots, N\,; \qquad c_N = -\frac{1}{2}\sum_{j=1}^{2N+1}\lambda_j\,. \tag{6}$$

The integrals of dp_N and dq_N over the α - cycles give the components of the wave number and the frequency vectors respectively

$$\oint_{\alpha_j} dp_N(\lambda) = k_j(\lambda_1, \ldots, \lambda_{2N+1})\,, \qquad \oint_{\alpha_j} dq_N(\lambda) = \omega_j(\lambda_1, \ldots, \lambda_{2N+1})\,, \qquad j = 1, \ldots, N\,. \tag{7}$$

Let $\lambda_1 = -1$, $\lambda_{2N+1} = 0$. Following Venakides [34] we introduce a lattice of points

$$1 \approx \eta_1 > \eta_2 > \ldots > \eta_N \approx 0\,, \tag{8}$$

where

$$-\eta_j^2 = \frac{1}{2}\,(\lambda_{2j-1} + \lambda_{2j}) \tag{9}$$

are the centres of bands.

We now assume that the spectral bands are distributed such that one can introduce two positive continuous functions on $[0,1]$:

1. The normalized density of bands $\varphi(\eta)$:

$$\varphi(\eta)d\eta \approx \frac{\text{number of lattice points in } (\eta,\ \eta + d\eta)}{N}\,.$$

That is,

$$\varphi(\eta_j) = \frac{1}{N(\eta_j - \eta_{j+1})} + O(\frac{1}{N})\,, \qquad \int_0^1 \varphi(\eta)d\eta = 1\,, \quad \eta^2 = -\lambda \in [0,1]\,. \tag{10}$$

2. The normalized logarithmic band width $\gamma(\eta)$:

$$\gamma(\eta_j) = -\frac{1}{N}\log\delta_j + O(\frac{1}{N})\,, \qquad \delta_j = \lambda_{2j} - \lambda_{2j-1}\,. \tag{11}$$

The functions $\varphi(\eta)$ and $\gamma(\eta)$ asymptotically define the *local* structure of the Riemann surface Γ (4) for $N \gg 1$. In other words, instead of $2N+1$ discrete parameters λ_j we have two continuous functions of η on $[0,1]$ which do not depend on x, t on the scale of the typical change of λ_j's in (3), say $\Delta x \sim \Delta t \sim l$.

The existence of the continuous distributions $\varphi(\eta)$ and $\gamma(\eta)$ implies the following band-gap scaling for $N \gg 1$:

$$|\mathrm{gap}_j| \sim \frac{1}{\varphi(\eta_j)N}\,, \qquad |\mathrm{band}_j| \sim \exp\{-\gamma(\eta_j)N\}\,, \quad j = 1, \dots, N \tag{12}$$

The scaling (12) has an important property: it preserves the finiteness of the integrated density of states as $N \to \infty$. The integrated density of states is defined in terms of the real part of the quasimomentum integral (see [21]):

$$\mathcal{N}_N(\lambda) = \frac{1}{\pi} Re \int\limits_{-1}^{\lambda} dp_N(\lambda')\,, \qquad \lambda \in [-1, 0]\,. \tag{13}$$

Now, using (7) one can readily see that

$$\mathcal{N}_N(\lambda) = \frac{1}{2\pi} \sum_{j=1}^{M} k_j \qquad \text{if} \quad \lambda \in [\lambda_{2M}, \lambda_{2M+1}]\,, \quad M = 1, \dots, N\,, \tag{14}$$

which is a particular (finite-gap) case of the general gap-labeling theorem for quasi-periodic potentials [21]. It is not difficult to show that the scaling (12) implies that $k_j \sim 1/N$ so the total density of states

$$\mathcal{N}_N(0) = \frac{1}{2\pi} \sum_{j=1}^{N} k_j \tag{15}$$

remains finite in the limit as $N \to \infty$. For this reason we shall call the limit as $N \to \infty$, defined on the spectral scaling (12), the *thermodynamic limit*.

We shall not be concerned here with the existence and the exact meaning of the thermodynamic limit for the finite-gap potentials $u_N(x,t)$ (which is a separate important problem) but shall rather directly consider this limit for the associated Whitham system (3). It is however, instructive to note that it follows from (12) that in the thermodynamic limit the band/gap ratio vanishes for each oscillating mode (i.e. $k_j \to 0\ \forall j = 1, 2\dots, N$), so the thermodynamic limit of the sequence of finite-gap potentials associated with the spectral scaling (12) is essentially an infinite-soliton limit. It was proposed in [6] that this limiting potential should be described in terms of ergodic random processes and can be viewed as a *homogeneous soliton gas* (or homogeneous soliton turbulence – depending on which of the two "identities" of a soliton is emphasized: the particle or the wave one). Then it is natural to assume that the same thermodynamic limit for the associated Whitham system should describe macroscopic evolution of the spatially *inhomogeneous* soliton gas. Indeed, as we shall see, the thermodynamic limit of the Whitham equations turns out to be consistent (in the small-density limit) with the kinetic equation for solitons derived by Zakharov [35] using the inverse scattering problem formalism.

We first note that $\mathcal{N}_N(\lambda)$ defined by (13) is a monotone increasing positive function so $d\mathcal{N}_N(\lambda)$ is a measure supported on the spectrum of the finite-gap potential $u_N(x)$ [21]. Next we introduce a 'temporal' analogue of the density of states (13) by the formula

$$\mathcal{V}_N(\lambda) = \frac{1}{\pi} Re \int\limits_{-1}^{\lambda} dq_N(\lambda')\,, \qquad \lambda \in [-1, 0]\,. \tag{16}$$

Then integration of the generating modulation equation (3) on the real axis of λ from -1 to $-\eta^2 \in [-1, 0]$ yields

$$\partial_t d\mathcal{N}_N(-\eta^2) = \partial_x d\mathcal{V}_N(-\eta^2)\,, \qquad \eta \in [0, 1]. \tag{17}$$

Thus the finite-gap Whitham-KdV system can be regarded as the system governing the evolution of the spectral measure.

Now we consider the thermodynamic limits of $d\mathcal{N}_N$ and $d\mathcal{V}_N$ which we denote as

$$d\mathcal{N}_N \to \pi f(\eta) d\eta\,, \qquad d\mathcal{V}_N \to -\pi f(\eta) s(\eta) d\eta \qquad \text{as} \quad N \to \infty, \tag{18}$$

where the limit is taken on the thermodynamic spectral scaling (12). Since $\pi f(\eta) d\eta$ is the limiting spectral measure, the function $f(\eta)$ has the natural meaning of the distribution function of the solitons over the spectrum (the meaning of the function $s(\eta)$ will become clear soon). The functions $f(\eta)$ and $s(\eta)$ were shown in [6], [5] to be expressed in terms of the ratio $\sigma(\eta) = \phi(\eta)/\gamma(\eta)$ of the lattice distribution functions (10), (11) by certain integral equations, which are then combined into a single equation directly connecting $f(\eta)$ and $s(\eta)$ [5]:

$$s(\eta) = 4\eta^2 + \frac{1}{\eta} \int\limits_0^1 \log \left| \frac{\eta + \mu}{\eta - \mu} \right| f(\mu)[s(\eta) - s(\mu)] d\mu\,. \tag{19}$$

We stress that in the continuum (thermodynamic) limit given by equations (18), (19) the explicit dependence of the density of states on the spectral branch points λ_j disappears. The only 'reminder' of the hyperelliptic Riemann surface Γ (4) is the kernel $\ln|\eta + \mu|/|\eta - \mu|$ which arises as the continuum limit of the off-diagonal elements of the period matrix $\mathbf{B}$ of the Riemann theta-function $\Theta_N(x, t|\mathbf{B})$ defining, via the Its-Matveev formula, the finite-gap potential (see [34] and [5]).

Thus, integral equation (19) can be viewed as a *local* (in the x, t - plane) relationship between the functions $f(\eta)$ and $s(\eta)$ characterizing the soliton gas. Let $l \gg 1$ be the characteristic length at which the change of functions $f(\eta)$, $s(\eta)$ is small (of an order $1/l \ll 1$). Next, in the spirit of modulation theory (see [33], [13]) we assume that on a larger spatiotemporal scale, $\Delta x \gg l$, $\Delta t \gg l$, we have $f(\eta) \equiv f(\eta, x, t)$, $s(\eta) \equiv s(\eta, x, t)$ and postulate, using (18), that

$$\partial_t d\mathcal{N}_N \to \pi \partial_t f(\eta, x, t) d\eta\,, \qquad \partial_x d\mathcal{V}_N \to -\pi \partial_x f(\eta, x, t) s(\eta, x, t) d\eta\,. \tag{20}$$

Then modulation equation (17) assumes the form of a conservation equation for f,

$$f_t + (sf)_x = 0\,, \tag{21}$$

which is clearly an expression of the isospectrality of the KdV evolution. Since $\rho(x, t) = \int_0^1 f d\eta$ is the density of solitons the quantity $s(\eta, x, t)$ can naturally be interpreted as the velocity of the soliton gas (or, more specifically, the velocity of a 'trial' soliton with the spectral parameter $\lambda = -\eta^2$ – see [20]). One can see from (19) that this velocity differs from the velocity $4\eta^2$ of the free soliton with the same spectral parameter. This difference is obviously due to the collisions of the 'trial' η-soliton with other 'μ' - solitons in the soliton gas. Indeed, for small densities $\rho = \int f d\eta \ll 1$ one can consider the second term in (19) as a small correction to the free-soliton velocity and obtain to the first order in ρ

$$s(\eta) \approx 4\eta^2 + \frac{1}{\eta} \int\limits_0^1 \ln \left| \frac{\eta + \mu}{\eta - \mu} \right| f(\mu)[4\eta^2 - 4\mu^2] d\mu \tag{22}$$

which is Zakharov's expression for the average velocity of a trial soliton in a rarefied soliton gas, obtained by taking into account the change in the soliton position due to phase shifts in its pairwise collisions with other solitons [35].

Equations (21) and (19) thus provide a self-consistent kinetic description of the KdV soliton gas of a finite density. We note that the upper limit in the integrals in (19), (22) can be replaced by $+\infty$ to make the kinetic equation independent on the original spectral lattice normalization (8).

The outlined procedure of the thermodynamic limit can be readily extended to the entire Whitham-KdV hierarchy,

$$(dp_N)_{t_n} = (dq_N^{(n)})_x\,, \quad n \in \mathbb{N}\,, \tag{23}$$

where n is a number of the "higher" Whitham-KdV equation in the hierarchy (the original modulation equation (3) corresponding to the KdV equation itself has the number $n = 1$) and t_n is the corresponding "higher" time, so that $(dp_N)_{t_n t_m} = (dp_N)_{t_m t_n}$ for all $n \neq m$. The meromorphic differential $dq_N^{(n)}$ is uniquely defined by its asymptotic behaviour near $\lambda = -\infty$,

$$dq^{(n)} \sim (-\lambda)^{n-1/2} d\lambda\,, \tag{24}$$

and the normalization

$$\oint\limits_{\beta_j} dq^{(n)}(\lambda) = 0\,, \qquad j = 1, \dots, N \tag{25}$$

analogous to (6).

Now, applying to equation (23) the outlined above procedure of the thermodynamic limiting transition we obtain the same transport equation (21) for the distribution function $f(\eta, x, t)$

$$f_{t_n} + (s_n f)_x = 0\,, \tag{26}$$

and the integral closure equation for s_n assumes the form

$$s_n(\eta) = C_n \eta^{2n} + \frac{1}{\eta}\int\limits_0^1 \log\left|\frac{\eta+\mu}{\eta-\mu}\right| f(\mu)[s_n(\eta) - s_n(\mu)]d\mu\,, \tag{27}$$

where C_n are certain constants. We won't need specific expressions for them here. Moreover, since the characteristic speeds of the commuting KdV-Whitham flows, and, therefore, the corresponding transport velocities s_n in the thermodynamic limit equation (27), are defined up to a constant factor, hereafter one can assume C_n to be arbitrary constants.

We note that equation(27) differs from (19) only in the first term corresponding to the free-soliton velocity. Also note that the 'phase-shift' logarithmic kernel in the integral equation (27) is the same for all n, which is not surprising as the whole finite-gap Whitham-KdV hierarchy (23) is associated with the same Riemann surface, i.e. with the same period matrix $\mathbf{B}$ responsible for the form of the integral kernel in the limit.

Now it is only natural to consider a generalization of the derived kinetic equations (26), (27) by introducing in (27) an arbitrary function $S(\eta)$ instead of the free-soliton velocity term and a symmetric function $G(\eta, \mu)$ instead of the 'phase-shift kernel' in the integral term. Also, for generality we replace the upper limit of integration in the closure equation (19) by $+\infty$. As a result, we arrive at the generalized kinetic equation (1), which will be our main concern in the remainder of the paper.

3 'Cold-gas' hydrodynamic reductions

We introduce in (1) $u(\eta, x, t) = \eta f(\eta, x, t)$, $v(\eta, x, t) = -s(\eta, x, t)$ and consider an N-component 'cold-gas' *ansatz*

$$u = \sum_{i=1}^{N} u^i(x,t)\delta(\eta - \eta_i)\,, \tag{28}$$

which reduces (1) to a system of hydrodynamic conservation laws,

$$u^i_t = (u^i v^i)_x, \qquad i = 1, \ldots, N\,, \tag{29}$$

where the velocities v^i and the 'densities' u^i satisfy algebraic relations

$$v^i = \xi_i + \sum_{m \neq i} \epsilon_{im} u^m (v^m - v^i), \qquad \epsilon_{ik} = \epsilon_{ki}. \tag{30}$$

Here

$$\xi_i = S(\eta_i)\,, \qquad \epsilon_{ik} = \frac{1}{\eta_i \eta_k} G(\eta_i, \eta_k)\,, \qquad v^i = v^i(x,t) = s(\eta_i, x, t) \tag{31}$$

Without loss of generality one may assume that $\epsilon_{ii} = 0$ for all i.

In a two-component case, the above algebraic system (30) can be easily resolved for $u^{1,2}$ in terms of $v^{1,2}$:

$$u^1 = \frac{1}{\epsilon_{12}} \frac{v^2 - \xi_2}{v^1 - v^2}, \qquad u^2 = \frac{1}{\epsilon_{12}} \frac{v^1 - \xi_1}{v^2 - v^1}. \tag{32}$$

Substituting (32) into (29) we arrive at the

Lemma 3.1 (El & Kamchatnov 2005 [7]): *Hydrodynamic type system (29), (30) for $N = 2$ reduces to a diagonal form in the field variables v^1 and v^2:*

$$v^1_t = v^2 v^1_x, \qquad v^2_t = v^1 v^2_x. \tag{33}$$

Remarkably, the hydrodynamic type system (33) is *linearly degenerate* because its characteristic velocities do not depend on the corresponding Riemann invariants.

It is worth noting that system (33) is equivalent to the 1D Born-Infeld equation (Born & Infeld 1934) arising in nonlinear electromagnetic field theory (see [33], [1])

$$(1 + \varphi_x^2)\varphi_{yy} - 2\varphi_x\varphi_y\varphi_{xy} + (1 - \varphi_y^2)\varphi_{xx} = 0.$$

As any two-component hydrodynamic type system, (33) is integrable (linearizable) via the classical hodograph transform. However, for any $N \geq 3$ integrability of the original system (29), (30) is no longer obvious. As a matter of fact, most N-component hydrodynamic type systems are *not integrable* for $N \geq 3$. Also, it is even not clear whether N-component system (29), (30) is linearly degenerate. It might seem that this system is simple enough for one to be able to verify these properties by a direct computation, using general definitions of linear degeneracy and integrability for hydrodynamic type systems [28], [31, 32] (also see the next section). To our surprise, even the simplest non-trivial case $N = 3$ turned out to be complicated enough to require computer algebra to get the confirmation of our hypothesis.

The identification of the system (29), (30) for $N = 3$ as an integrable linearly degenerate hydrodynamic system can be considered as a strong indication that both properties (linear degeneracy and integrability) could hold true for this system for arbitrary N. Thus we formulate our main

Theorem 3.1 *N-component reductions (29), (30) of the generalized kinetic equation (1) are linearly degenerate integrable hydrodynamic type systems for any N.*

To prove this theorem, we take advantage of the well-developed theory of integrable linearly degenerate hydrodynamic type systems [28], [9].

4 Linearly degenerate integrable hydrodynamic type systems: account of properties

In this Section we give a brief account of the general theory of semi-Hamiltonian linearly degenerate hydrodynamic type systems.

A hydrodynamic type system

$$U_t^i = v_j^i(\mathbf{U})U_x^j, \quad i = 1, 2, ..., N, j = 1, 2, ..., N \tag{34}$$

is called semi-Hamiltonian (see [31, 32]) if it

(i) has N mutually distinct eigenvalues $\lambda^i(\mathbf{U})$

$$\det\left|\lambda\delta_j^i - v_j^i(\mathbf{U})\right| = 0; \tag{35}$$

(ii) admits invertible point transformations $U^k(\mathbf{r})$, such that this hydrodynamic type system can be written in diagonal form

$$r_t^i = V^i(\mathbf{r})r_x^i, \qquad i = 1, \ldots, N. \tag{36}$$

The variables $r^k(\mathbf{U})$ are called Riemann invariants and $V^k(\mathbf{r}) = \lambda^k(\mathbf{U}(\mathbf{r}))$ – characteristic velocities. Each Riemann invariant r^i is determined up to an arbitrary function of a single variable $R_i(r^i)$.

(iii) satisfies the identity

$$\partial_j \frac{\partial_k V^i}{V^k - V^i} = \partial_k \frac{\partial_j V^i}{V^j - V^i}, \quad i \neq j \neq k \tag{37}$$

for each three distinct characteristic velocities ($\partial_k \equiv \partial/\partial r^k$).

A semi-Hamiltonian hydrodynamic type system possesses infinitely many conservation laws parameterised by N arbitrary functions of a single variable. Its general local solution for $\partial_x r^i \neq 0$, $i = 1, \ldots, N$ is given by the generalised hodograph formula [31, 32]

$$x + V^i(\mathbf{r})t = W^i(\mathbf{r})\,, \tag{38}$$

where functions $W^i(\mathbf{r})$ are found from the linear system of PDEs:

$$\frac{\partial_i W^j}{W^i - W^j} = \frac{\partial_i V^j}{V^i - V^j}\,, \quad i, j = 1, \ldots, N, \quad i \neq j. \tag{39}$$

Thus, the semi-Hamiltonian property (37) implies integrability of diagonal hydrodynamic type system in the above generalised hodograph sense.

It is not difficult to verify that commuting hydrodynamic flows of (36) are specified by the hydrodynamic type system

$$r^j_\tau = W^j(\mathbf{r}) r^j_x\,, \quad j = 1, \ldots, N\,, \tag{40}$$

where τ is a new time (group parameter) so that $(r^j_\tau)_t = (r^j_t)_\tau$ implies (39).

A sub-class of linearly degenerate hydrodynamic type systems is determined by the property

$$\partial_i V^i = 0 \tag{41}$$

for each index i. It means that each characteristic velocity does not depend on the corresponding Riemann invariant r^i.

Theorem 4.1 (Pavlov 1987 [28]): *If semi-Hamiltonian hydrodynamic type system (36) possesses conservation laws (29) with $u^i = U^i(\mathbf{r})$ and $v^i(\mathbf{U}(\mathbf{r})) = V^i(\mathbf{r})$ then this system is linearly degenerate. These conservation laws (29) are parameterized by N arbitrary functions of a single variable.*

Proof: The semi-Hamiltonian property (i.e. *integrability*) is given by the condition (37). We introduce, following Tsarev [32], the so-called Lame coefficients $\bar{H}_i$ by

$$\partial_k \ln \bar{H}_i = \frac{\partial_k V^i}{V^k - V^i}, \quad i \neq k. \tag{42}$$

Suppose that some semi-Hamiltonian hydrodynamic type system (36) can be written in the conservative form (29) with $v^i(\mathbf{U}(\mathbf{r})) = V^i(\mathbf{r})$. In such a case

$$\partial_k U^i \cdot r^k_t = \partial_k (U^i V^i) \cdot r^k_x.$$

Since $\mathbf{r}(x, t)$ is an arbitrary solution of (36) we obtain N equations

$$V^k \cdot \partial_k U^i = \partial_k (U^i V^i). \tag{43}$$

If $k \neq i$, then

$$\partial_k \ln U^i = \frac{\partial_k V^i}{V^k - V^i}, \tag{44}$$

i.e. each of the conservation law densities U^i is determined up to an arbitrary function of a single variable $P_i(r^i)$ (cf. (42) and (44)),

$$U^i = \bar{H}_i \cdot P_i(r^i). \tag{45}$$

If $k = i$, then it follows from (43) that $\partial_i V^i = 0$ i.e the system is linearly degenerate. The theorem is proved.

Remark: A subset $\{u^k\}$ of the conservation law densities $\{U^k\}$ satisfying a given system of conservation laws (e.g. (29), (30)) is selected by fixing the functions P_k (e.g. $P_k(r^k) \equiv 1$ — see (91) in Section 7).

It is worth noting that the statement converse to Theorem 4.1 is not valid in general. Indeed, let us consider the two-component system of conservation laws,

$$U^1_t = (U^1 v^1(U^1, U^2))_x, \qquad U^2_t = (U^2 v^2(U^1, U^2))_x. \tag{46}$$

Suppose this hydrodynamic type system is linearly degenerate, then it can be written in Riemann invariants $r^1(U^1, U^2)$, $r^2(U^1, U^2)$ as follows:

$$r_t^1 = V^1(r^1, r^2) r_x^1, \quad r_t^2 = V^2(r^1, r^2) r_x^2,$$

where $V^{1,2}(\mathbf{r}) = v^{1,2}(\mathbf{U}(\mathbf{r}))$. Let us introduce new conservation law densities $\tilde{U}^1 = U^1 + U^2$ and $\tilde{U}^2 = U^1 - U^2$. Then the system of conservation laws (46) assumes an equivalent form

$$\tilde{U}_t^1 = (\tilde{U}^1 \tilde{v}^1(\tilde{U}^1, \tilde{U}^2))_x, \quad \tilde{U}_t^2 = (\tilde{U}^2 \tilde{v}^2(\tilde{U}^1, \tilde{U}^2))_x,$$

where the characteristic velocities

$$\tilde{v}^1 = \frac{U^1 v^1 + U^2 v^2}{U^1 + U^2}, \quad \tilde{v}^2 = \frac{U^1 v^1 - U^2 v^2}{U^1 - U^2}$$

no longer coincide with $V^1(r^1, r^2)$ and $V^2(r^1, r^2)$.

The full theory of linearly degenerate semi-Hamiltonian hydrodynamic type systems was constructed by Ferapontov in [9] using the Stäckel matrices

$$\Delta = \begin{pmatrix} \phi_1^1(r^1) & ... & \phi_N^1(r^N) \\ & & \\ ... & ... & ... \\ & & \\ \phi_1^{N-2}(r^1) & & \phi_N^{N-2}(r^N) \\ \phi_1^{N-1}(r^1) & & \phi_N^{N-1}(r^N) \\ 1 & ... & 1 \end{pmatrix} \tag{47}$$

where $\phi_k^i(r^k)$ are arbitrary functions (it is clear that without loss of generality one can put $\phi_k^{N-1}(z) = z$ and the number of arbitrary function reduces to $N(N-2)$). Then characteristic velocities of such linearly degenerate hydrodynamic type systems are given by the formula

$$V^i = \frac{\det \Delta_i^{(2)}}{\det \Delta_i^{(1)}}, \tag{48}$$

where $\Delta_i^{(k)}$ is the matrix Δ without kth row and ith column. The family of the conservation law densities U^i corresponding to the semi-Hamiltonian system (36), (48) is determined by (cf. (45))

$$U^i = \frac{\det \Delta_i^{(1)}}{\det \Delta} (-1)^{i+1} P_i(r^i), \tag{49}$$

where $P_i(r^i)$, $i = 1, \ldots, N$ are arbitrary functions.

Corollary 4.1 *The system of conservation laws (29) is a semi-Hamiltonian linearly degenerate hydrodynamic type system if and only if the densities u^i and velocities $v^i(\mathbf{u})$ admit representations $u^i = U^i(\mathbf{r})$ and $v^i(\mathbf{U}(\mathbf{r})) = V^i(\mathbf{r})$, specified by (49), (48), via N functions r^k. The functions $r^k(x, t)$ then satisfy diagonal system (36).*

Proposition 4.1 (Ferapontov 1991 [9]): *Semi-Hamiltonian linearly degenerate hydrodynamic type system (36), (48) has $N-2$ nontrivial linearly degenerate commuting flows*

$$r^j_{t^k} = V^j_{(k)}(\mathbf{r}) r^j_x, \quad j = 1, \dots, N\,, \quad k = 3, 4, ..., N, \tag{50}$$

whose characteristic velocities are determined as (cf. (48))

$$V^i_{(k)} = \frac{\det \Delta^{(k)}_i}{\det \Delta^{(1)}_i}\,. \tag{51}$$

Any characteristic velocity vector $\mathbf{W}(\mathbf{r}) = (W^1(\mathbf{r}), W^2(\mathbf{r}), \dots, W^N(\mathbf{r}))$ specifying linearly degenerate hydrodynamic flow (40) commuting with (36), (48), can be represented as a linear combination of the "basis" characteristic velocity vectors $\mathbf{V}_{(k)}$ (51) (including "trivial" ones $\mathbf{V}_{(2)} \equiv \mathbf{V}$ (see (48)) and $\mathbf{V}_{(1)} \equiv \mathbf{1}$) with some constant coefficients b_k. Thus, for any component W^i there exists a decomposition

$$W^i = \sum_{k=1}^{N} b_k V^i_{(k)}\,. \tag{52}$$

Theorem 4.2 (Ferapontov 1991 [9]): *General solution $\mathbf{r}(x,t)$ of the semi-Hamiltonian linearly degenerate system (36) is parameterized by N arbitrary functions of one variable $f_k(r^k)$ and is given in an implicit form by the algebraic system*

$$\begin{aligned} x &= \sum_{k=1}^{N} \int^{r^k} \frac{\phi^1_k(\xi) d\xi}{f_k(\xi)}, \qquad -\,t = \sum_{k=1}^{N} \int^{r^k} \frac{\phi^2_k(\xi) d\xi}{f_k(\xi)} \\ 0 &= \sum_{k=1}^{N} \int^{r^k} \frac{\phi^M_k(\xi) d\xi}{f_k(\xi)}, \qquad M = 3, 4, ..., N. \end{aligned} \tag{53}$$

(note the change of sign for t compared to [9] due to a slightly different representation of the diagonal system (36) in this paper). We also note that formulae (53) represent an equivalent of the symmetric generalised hodograph solution (38) for semi-Hamiltonian linearly degenerate hydrodynamic type systems.

It is instructive to introduce, following Darboux [8], the so-called rotation coefficients

$$\beta_{ik} = \frac{\partial_i \bar{H}_k}{\bar{H}_i}, \quad i \neq k\,, \tag{54}$$

where the Lamé coefficients $\bar{H}_i$ are defined by (42). Then expression (37) for the semi-Hamiltonian property assumes the form

$$\partial_i \beta_{jk} = \beta_{ji}\beta_{ik}, \quad i \neq j \neq k\,. \tag{55}$$

Using (54) linear system (39) can be related to another linear system

$$\partial_i H_k = \beta_{ik} H_i, \quad i \neq k, \tag{56}$$

via the so-called Combescure transformation (see [8])

$$W^i = \frac{H_i}{\bar{H}_i}. \tag{57}$$

In other words, one can show (see [32]) that the ratio of any two solutions to (56) satisfies system (39) for the characteristic velocities of the commuting flows (40). We note that the particular solution $\tilde{H}_i$ of (56) corresponding to the characteristic velocities V_i of the original system (36) is expressed in terms of the Lamé coefficient $\bar{H}_i$ as

$$\tilde{H}_i = V^i \bar{H}_i. \tag{58}$$

Of course, general solution H_i of system (56), as well as general solution W^i of the generalized hodograph equations (39), is parameterized by N arbitrary functions of single variable.

Theorem 4.3 (Pavlov 1987 [28]): *The class of the semi-Hamiltonian linearly degenerate systems of hydrodynamic type is selected, in addition to (55), by the set of restrictions on the rotation and Lame coefficients*

$$\partial_i \ln \bar{H}_i, = \partial_i \ln \beta_{ji} \tag{59}$$

for any index $j \neq i$.

Proof : Let us consider the Lamé coefficients for the linearly degenerate systems. Using (42), (41) we have

$$\partial_j V^i = \partial_j \ln \bar{H}_i \cdot (V^j - V^i), \quad i \neq j, \qquad \partial_i V^i = 0.$$

The compatibility condition $\partial_i(\partial_j V^i) = \partial_j(\partial_i V^i)$ implies that

$$\partial_i \partial_j \ln \bar{H}_i = \partial_j \ln \bar{H}_i \cdot \partial_i \ln \bar{H}_j, \quad i \neq j. \tag{60}$$

Now one can see that the l.h.s. of (60) can be written in the form

$$\partial_i \partial_j \ln \bar{H}_i = \partial_i \left(\frac{\bar{H}_j}{\bar{H}_i} \beta_{ji} \right) = \beta_{ij}\beta_{ji} + \frac{\bar{H}_j}{\bar{H}_i} \partial_i \beta_{ji} - \frac{\bar{H}_j}{\bar{H}_i^2} \beta_{ji} \partial_i \bar{H}_i. \tag{61}$$

On the other hand, the r.h.s. of (60) is nothing but the product $\beta_{ij}\beta_{ji}$. Now (59) immediately follows from (60) and (61). The Theorem is proved.

Now, suppose that the rotation coefficients (54) for some linearly degenerate hydrodynamic type system are given. Then restrictions (59) determine not only the Lamé coefficients (42) but also all other solutions of (56) associated, via (57), with the characteristic velocities (48), (51) of the complete set of linearly degenerate commuting flows. Indeed, one can see that equations (54), (59) actually represent N systems *ordinary differential equations* so that each system contains differentiation with respect to only one Riemann invariant. Thus, the general solution $\bar{H}_i$ of system (54), (59) is parameterized by N arbitrary constants (see Proposition 4.1).

Let us introduce N particular solutions $\bar{H}_i^{(k)}$ of system (54), (59) such that (see (48), (51))

$$V^i_{(k)} = \frac{\bar{H}_i^{(k)}}{\bar{H}_i}, \quad k = 2, 3, ..., N,$$

where $\bar{H}_i = \bar{H}_i^{(1)}, \tilde{H}_i = \bar{H}_i^{(k)}$ (see (51)). As a matter of fact, $V^i_{(2)} \equiv V^i, V^i_{(1)} \equiv 1$. Then (59) can be written in a slightly more general form,

$$\partial_i \ln \beta_{ji} = \partial_i \ln \bar{H}_i^{(k)} ,$$

– for any k and $j \neq i$.

Thus, the full class of linearly degenerate semi-Hamiltonian hydrodynamic type systems is determined by conditions (59), (54) and (55). We note that system (59), (54) and (55) is an overdetermined system in involution. Its integration leads to the aforementioned set of particular solutions of (56) that can be parameterized via a Stäckel matrix (see (47), (48), (49) and (51)).

5 $N = 3$: explicit formulae

We now apply the construction outlined in the previous Section to the first nontrivial (from the viewpoint of integrability) case $N = 3$ of the hydrodynamic reduction (29), (30). To prove our main Theorem 3.1 for $N = 3$ we make use of Corollary 3.1.

Let us suppose that hydrodynamic system of conservation laws (29), (30) is linearly degenerate and can be written in a diagonal form (36), i.e. we suppose that there exists an invertible change of variables $r^j(\mathbf{u})$, $j = 1, 2, 3$, such that system (29) assumes a diagonal form

$$r^j_t = V^j(\mathbf{r}) r^j_x , \qquad j = 1, 2, 3, \tag{62}$$

where $V^j(\mathbf{r}) = v^j(\mathbf{u}(\mathbf{r}))$.

We introduce the Stäckel matrix (47), which for $N = 3$ can be written in the form

$$\Delta = \begin{pmatrix} B_1(r^1) & B_2(r^2) & B_3(r^3) \\ A_1(r^1) & A_2(r^2) & A_3(r^3) \\ 1 & 1 & 1 \end{pmatrix} , \tag{63}$$

where $A_k(z)$ and $B_k(z)$ are arbitrary functions.

Now, by Corollary 3.1, if system (29), (30) is linearly degenerate and semi-Hamiltonian then its diagonal representation (62) must have characteristic velocities in the form (48), i.e. for $N = 3$ we have

$$V^1 = \frac{B_2(r^2) - B_3(r^3)}{A_2(r^2) - A_3(r^3)}, \quad V^2 = \frac{B_3(r^3) - B_1(r^1)}{A_3(r^3) - A_1(r^1)}, \quad V^3 = \frac{B_1(r^1) - B_2(r^2)}{A_1(r^1) - A_2(r^2)}. \tag{64}$$

Then, using (49) the corresponding conservation law densities u^k are found in terms of Riemann invariants as

$$u^1 = \frac{P_1(r^1)}{\det \Delta}[A_2(r^2) - A_3(r^3)], \quad u^2 = \frac{P_2(r^2)}{\det \Delta}[A_3(r^3) - A_1(r^1)], \quad u^3 = \frac{P_3(r^3)}{\det \Delta}[A_1(r^1) - A_2(r^2)], \tag{65}$$

where $P_j(r^j)$ are arbitrary functions and the determinant of the Stäckel matrix is given by

$$\det \Delta = A_1(r^1)[B_2(r^2) - B_3(r^3)] + A_2(r^2)[B_3(r^3) - B_1(r^1)] + A_3(r^3)[B_1(r^1) - B_2(r^2)]. \tag{66}$$

Substitution of (64)–(66) into (30) yields expressions for the functions $A_k(z)$, $B_k(z)$, $P_k(z)$, $k = 1, 2, 3$.

Before we present these expressions, we note that it follows from (64), (66) that functions $B_k(z)$ are determined up to a constant shift which is then translated into a certain shift for the functions $P_k(z)$. It turns out that this shift can be removed by the simplest change of the Riemann invariants, $(r^k + \text{constant}) \mapsto r^k$ (although the relationships between the shift constants for B_k, P_k and r^k are rather cumbersome) so that we eventually obtain

$$A_i(r^i) = r^i \,, \qquad B_i(r^i) = \zeta_i r^i \,, \quad i = 1, 2, 3, \tag{67}$$

where

$$\zeta_1 = \frac{\xi_3\epsilon_{12} - \xi_2\epsilon_{13}}{\epsilon_{12} - \epsilon_{13}} \,, \quad \zeta_2 = \frac{\xi_1\epsilon_{23} - \xi_3\epsilon_{12}}{\epsilon_{23} - \epsilon_{12}} \,, \quad \zeta_3 = \frac{\xi_1\epsilon_{23} - \xi_2\epsilon_{13}}{\epsilon_{23} - \epsilon_{13}} \,, \tag{68}$$

$$P_1 = \frac{\xi_2 - \xi_3}{\epsilon_{12} - \epsilon_{13}} r^1 + \epsilon_{23} \,, \quad P_2 = \frac{\xi_1 - \xi_3}{\epsilon_{12} - \epsilon_{23}} r^2 + \epsilon_{13} \,, \quad P_3 = \frac{\xi_1 - \xi_2}{\epsilon_{13} - \epsilon_{23}} r^3 + \epsilon_{12} \,. \tag{69}$$

Direct verification shows that the diagonal system (62), (64), (67), (68) is indeed equivalent, via (65), (66), (69), to the original set of conservation laws (29), (30), where $v^k(\mathbf{u}(\mathbf{r})) = V^k(\mathbf{r})$.

Thus, system (29), (30) is consistent with formulae (64), (65) defined by Stäckel matrix (63). Therefore, by Corollary 3.1, the three-component hydrodynamic reduction (29), (30) is a linearly degenerate semi-Hamiltonian (i.e. integrable) hydrodynamic type system.

Remark: As we have seen, the outlined construction has some additional inherent "degrees of freedom", namely, three arbitrary constants due to non-uniqueness of the Stäckel matrix specifying a given linearly degenerate semi-Hamiltonian system. The full set of arbitrary constants removable by an appropriate change of the Riemann invariants will appear later in Section 5 where we shall consider N-component hydrodynamic reductions with arbitrary $N \geq 3$.

Using (64)–(69) we obtain explicit expressions for the characteristic velocities V^k and densities u^k in terms of Riemann invariants,

$$V^1 = \frac{\zeta_2 r^2 - \zeta_3 r^3}{r^2 - r^3}, \quad V^2 = \frac{\zeta_3 r^3 - \zeta_1 r^1}{r^3 - r^1}, \quad V^3 = \frac{\zeta_1 r^1 - \zeta_2 r^2}{r^1 - r^2}, \tag{70}$$

$$u^1 = P_1 \frac{r^2 - r^3}{\det \Delta}, \qquad u^2 = P_2 \frac{r^3 - r^1}{\det \Delta}, \qquad u^3 = P_3 \frac{r^1 - r^2}{\det \Delta}, \tag{71}$$

where

$$\det \Delta = (\zeta_1 - \zeta_2) r^1 r^2 + (\zeta_2 - \zeta_3) r^2 r^3 + (\zeta_3 - \zeta_1) r^3 r^1 \,. \tag{72}$$

We note that, unlike in the case $N = 2$, algebraic system (30) cannot be resolved for u^k in terms of v^n for any odd N (cf. corresponding formulae in Section 2), because determinant of the matrix $\hat{\mathbf{A}}$ of linear system (30)

$$\hat{\mathbf{A}}\mathbf{u} = \mathbf{b},$$

where $A_{ik} = \epsilon_{ik}(v^k - v^i)$ and $b_i = v^i - \xi_i$, equals zero due to its skewsymmetry. For instance, for $N = 3$, the consistency condition of this linear system (i.e. the condition that the rank of augmented matrix equals 2) is given by the relation

$$\epsilon_{23}(v^3 - v^2)(\xi_1 - v^1) + \epsilon_{12}(v^2 - v^1)(\xi_3 - v^3) + \epsilon_{13}(v^1 - v^3)(\xi_2 - v^2) = 0. \tag{73}$$

Direct substitution of $v^j = V^j(\mathbf{r})$ (70) into (73) shows that it satisfies identically.

Using (71), (72), (68), (69) one can express the Riemann invariants in terms of the densities u^k explicitly,

$$\begin{aligned} r^1 &= \frac{(\epsilon_{12}-\epsilon_{13})(\epsilon_{12}\epsilon_{13}u^1+\epsilon_{12}\epsilon_{23}u^2+\epsilon_{13}\epsilon_{23}u^3+\epsilon_{23})}{[(\xi_3-\xi_1)\epsilon_{12}+(\xi_1-\xi_2)\epsilon_{13}]u^1-(\xi_2-\xi_3)(\epsilon_{12}u^2+\epsilon_{13}u^3+1)}, \\ r^2 &= \frac{(\epsilon_{23}-\epsilon_{12})(\epsilon_{12}\epsilon_{13}u^1+\epsilon_{12}\epsilon_{23}u^2+\epsilon_{13}\epsilon_{23}u^3+\epsilon_{13})}{[(\xi_1-\xi_2)\epsilon_{23}+(\xi_2-\xi_3)\epsilon_{12}]u^2-(\xi_3-\xi_1)(\epsilon_{12}u^1+\epsilon_{23}u^3+1)}, \\ r^3 &= \frac{(\epsilon_{13}-\epsilon_{23})(\epsilon_{12}\epsilon_{13}u^1+\epsilon_{12}\epsilon_{23}u^2+\epsilon_{13}\epsilon_{23}u^3+\epsilon_{12})}{[(\xi_2-\xi_3)\epsilon_{13}+(\xi_3-\xi_1)\epsilon_{23}]u^3-(\xi_1-\xi_2)(\epsilon_{13}u^1+\epsilon_{23}u^2+1)}. \end{aligned} \tag{74}$$

Direct substitution shows that expressions (74) and (70) are consistent with original algebraic system (30) where $v^j = V^j(\mathbf{r}(\mathbf{u}))$.

It is instructive to look at what happens to the diagonal system (62) when the density of one of the components in conservation laws (29), say u^3, vanishes. One can see from (74) that if $u^3 = 0$ (this corresponds to vanishing of P_3 in (71)) then the Riemann invariant r^3 becomes a constant,

$$u^3 = 0: \qquad r^3 = -\frac{(\epsilon_{23}-\epsilon_{13})\epsilon_{12}}{\xi_1-\xi_2} \equiv r^3_0\,,$$

so that the equation for r^3 satisfies identically and system (62) reduces to its 2-component counterpart (33) for

$$v^1(u^1,u^2) = V^1(r^2(u^1,u^2,0))\,, \qquad v^2(u^1,u^2) = V^2(r^1(u^1,u^2,0))\,,$$

as one should expect. Similar reductions occur for $u^1 = 0$ and $u^2 = 0$, which lead to $r^1 = r^1_0 =$ constant and $r^2 = r^2_0 =$ constant respectively. As a matter of fact, any function $R^j(r^j)$ is also a Riemann invariant so one can choose a new set of Riemann invariants say $R^j = r^j - r^j_0$ so that $R^j = 0$ when $u^j = 0$, which could be useful for applications.

Now we consider some special families of solutions to the linearly degenerate system (62), (70).

a) Similarity solutions

One can see that, owing to homogeneity of the characteristic velocities (70) as functions of Riemann invariants, system (62) admits similarity solutions of the form

$$r^i = \frac{1}{t^\alpha} l^i\left(\frac{x}{t}\right), \quad i = 1,2,3, \tag{75}$$

where α is an arbitrary real number and the functions $l^i(\tau)$, satisfy the system of ordinary differential equations

$$(V^i(\mathbf{l}) + \tau)\frac{dl^i}{d\tau} + \alpha l^i = 0\,, \quad i = 1,2,3. \tag{76}$$

Here the functions $V^i(\mathbf{l})$ are obtained from (70) by replacing r^i with l^i. It is not difficult to see that, due to the structure of the characteristic velocities, the case $\alpha = 0$ implies only a constant solution $l^i = l^i_0$, where l^1_0, l^2_0, l^3_0 are arbitrary constants. If $\alpha \neq 0$, the general solution of (76) can be found in an implicit form using the generalised hodograph formulae (53), where for $N = 3$ we substitute, according to (63), (67), $\phi^1_k(\xi) \equiv B_k(\xi) = \zeta_k \xi$, $\phi^2_k(\xi) \equiv A_k(\xi) = \xi$.

To obtain similarity solutions (75) one should use in (53) $f_i(\xi) = \xi^\beta/c_i$, where $\beta = 2 + 1/\alpha$ and c_i, $i = 1, 2, 3$, are arbitrary nonzero constants. Then the requirement that the functions l^i must depend on $\tau = x/t$ alone leads to the algebraic system

$$
\begin{aligned}
\tau &= c_1\zeta_1(l^1)^\gamma + c_2\zeta_2(l^2)^\gamma + c_3\zeta_3(l^3)^\gamma\,, \\
-1 &= c_1(l^1)^\gamma + c_2(l^2)^\gamma + c_3(l^3)^\gamma\,, \qquad (77)\\
0 &= c_1(l^1)^{\gamma-1} + c_2(l^2)^{\gamma-1} + c_3(l^3)^{\gamma-1}\,,
\end{aligned}
$$

where $\gamma = -1/\alpha$ and we have also replaced $c_i/\gamma \mapsto c_i$. Direct substitution shows that solution l^i defined (77) indeed satisfies system (76).

b) Quasiperiodic solutions

Another interesting type of solutions arises when one introduces in (53) (for $N = 3$)

$$
f_1(\xi) = f_2(\xi) = f_3(\xi) = \sqrt{R_7(\xi)}\,, \qquad R_7(\xi) = \prod_{m=1}^{7}(\xi - E_m)\,,
$$

where $E_1 < E_2 < \cdots < E_7$ are real constants. Then, according to (67), solution (53) assumes the form

$$
x = \zeta_1\int^{r^1}\frac{\xi d\xi}{\sqrt{R_7(\xi)}} + \zeta_2\int^{r^2}\frac{\xi d\xi}{\sqrt{R_7(\xi)}} + \zeta_3\int^{r^3}\frac{\xi d\xi}{\sqrt{R_7(\xi)}}, \qquad (78)
$$

$$
-t = \int^{r^1}\frac{\xi d\xi}{\sqrt{R_7(\xi)}} + \int^{r^2}\frac{\xi d\xi}{\sqrt{R_7(\xi)}} + \int^{r^3}\frac{\xi d\xi}{\sqrt{R_7(\xi)}}, \qquad (79)
$$

$$
0 = \int^{r^1}\frac{d\xi}{\sqrt{R_7(\xi)}} + \int^{r^2}\frac{d\xi}{\sqrt{R_7(\xi)}} + \int^{r^3}\frac{d\xi}{\sqrt{R_7(\xi)}}, \qquad (80)
$$

which resembles the celebrated system for the multi-gap (here – three-gap) solutions of the KdV equation. Unlike (78) - (80), however, the three-gap KdV solutions correspond to the Stäckel matrix (63) with the rows $A_k(\xi) = \xi$, $B_k(\xi) = \xi^2$, $k = 1, 2, 3$ [9].

Proposition 5.1. *For any constants $\zeta_1 \neq \zeta_2 \neq \zeta_3 \neq 0$ there exists at least one set $\{E_1, \dots, E_6\}$ such that the solution $r^i(x,t)$, $i = 1, 2, 3$ described by (78) - (80) is quasi-periodic in x and possibly in t.*

We present here a sketch of the proof. Availability of the solution in the form (78) - (80) implies the existence of separate dynamics of r^j-s with respect to x and t. Indeed, differentiating (78) - (80) with respect to x for fixed t one readily obtains

$$
\frac{\partial r^i}{\partial x} = (r^j - r^k)\frac{\sqrt{R_7(r_i)}}{\Pi}\,, \qquad i, j, k = 1, 2, 3, \quad i \neq j \neq k\,, \qquad (81)
$$

where

$$
\Pi(r_1, r_2, r_3) = (\zeta_1 - \zeta_2)r^1r^2 + (\zeta_2 - \zeta_3)r^2r^3 + (\zeta_3 - \zeta_1)r^3r^1 = \det\Delta \qquad (82)
$$

– see (72).

Analogously, differentiating (78) - (80) with respect to t for fixed x one obtains

$$\frac{\partial r^i}{\partial t} = (\zeta_j r^j - \zeta_k r^k)\frac{\sqrt{R_7(r_i)}}{\Pi}, \qquad i,j,k = 1,2,3, \quad i \neq j \neq k\,. \tag{83}$$

One can see that the flows (81) and (83) are consistent with the spatio-temporal dynamics (36), (70). We also note that equations (81), (83) resemble Dubrovin's equations for the auxiliary spectrum dynamics in the KdV finite-gap integration problem (see, for instance, [26]).

Let us now suppose that

$$r^1 \in [E_1, E_2],\ r^2 \in [E_3, E_4],\ r^3 \in [E_5, E_6], \tag{84}$$

so that all $\sqrt{R_7(r^i)}$ are real. The above condition (84) means that the point $p = (r^1, r^2, r^3) \in \mathbb{R}^3$ lies within the rectangular box $K_{ijk} \in \mathbb{R}^3$ with the vertices at (E_i, E_j, E_k), $i,j,k = 1,\dots,6$, $i \neq j \neq k$.

Now, for any set of the constants $\zeta_1, \zeta_2, \zeta_3$ there exists at least one box $K_{i,j,k} = K^* \in \mathbb{R}^3$, which is not intersected by the cone $\Pi(r^1, r^2, r^3) = 0$. That is, inside K^* the denominator $\Pi(r_1, r_2, r_3)$ in (81) never vanishes.

Assume now that the 'initial' values of r^1, r^2, r^3 for some $x = x_0$ belong to K^*. Then it follows from (81) that, under the x-flow ($t =$ const), the point p remains inside K^* and undergoes "elastic" reflections at the faces of K^* as x varies (note that, since $r^j \neq r^k$ for $j \neq k$, the factor $(r^j - r^k)$ in (81) never vanishes so the reflections occur only at the faces of K^*). Therefore, the motion is quasi-periodic with respect to x as long as conditions (84) are satisfied. Indeed, the system (81) possesses two integrals (79) and (80) outside the "resonant" points, where $\Pi = 0$, so it specifies a quasi-periodic motion on a 3-torus provided conditions (84) are satisfied. Of course, if conditions (84) are not satisfied at $x = x_0$ the solutions $r^i(x)$ may blow up and not be quasi-periodic.

The proof of quasi-periodicity of the t-flow is similar, however, there is an additional requirement that the factor $(\zeta_j r^j - \zeta_k r^k)$ in (82) should not vanish for all $\mathbf{r} \in K^*$ which might impose additional restrictions on the choice of E_i (that is for some $\{E_j\}$ the motion can be quasi-periodic in x but not in t).

We note that the quasi-periodicity of the x- and t-flows can be proved directly from the solution (78) – (80), however the outlined proof using the dynamical systems arguments is qualitatively more transparent and more readily yields the "resonant" restrictions for x- and t-flows.

6 Integrability of N-component hydrodynamic reductions

Now we prove our main Theorem 3.1 stating that the N-component 'cold-gas' hydrodynamic reduction (29), (30) represents a semi-Hamiltonian (i.e. integrable) linearly degenerate hydrodynamic type system. For that, according to Corollary 3.1, it is sufficient to show that the conservation law densities u^i and the transport velocities v^i admit parametric representations (49) and (48), $u^i = U^i(\mathbf{r})$ and $v^i(\mathbf{U}(\mathbf{r})) = V^i(\mathbf{r})$, via N functions r^k in terms of the Stäckel matrix (47).

We suppose that hydrodynamic type system (29), (30) can be rewritten in a diagonal form (36), and, moreover, the characteristic velocities $V^i(\mathbf{r})$ coincide with the expressions $v^i(\mathbf{U}(\mathbf{r}))$.

Now, substitution of (48), (49) into (30) leads to the algebraic system

$$\sum_{k=1}^{N} \epsilon_{ik}(-1)^k P_k \det \Delta_{ik}^{(12)} = \det \Delta_i^{(2)} - \xi_i \det \Delta_i^{(1)}, \qquad i = 1, \ldots, N\,, \tag{85}$$

for $P_k(r^k)$ and $\phi_k^i(r^k)$. Here the matrix $\Delta_{ik}^{(12)}$ is the matrix Δ with *first two* rows and ith and kth columns deleted. In the derivation of (85) we have used the *determinant Sylvester identity* (see, for instance, Gantmaher 1959)

$$\det \Delta_{ik}^{(12)} = \frac{\det \Delta_k^{(1)} \det \Delta_i^{(2)} - \det \Delta_i^{(1)} \det \Delta_k^{(2)}}{\det \Delta}\,.$$

Expanding the determinants,

$$\det \Delta_i^{(1)} = \sum_{k=1}^{N} \left[(-1)^{k+1} \phi_k^2 \det \Delta_{ik}^{(12)}\right], \qquad \det \Delta_i^{(2)} = \sum_{k=1}^{N} \left[(-1)^{k+1} \phi_k^1 \det \Delta_{ik}^{(12)}\right],$$

we rewrite equations (85) as N *nonlinear* systems for ϕ_k^n *and* P_k, where $k, n = 1, \ldots, N$,

$$\sum_{k=1}^{N} (-1)^k (\phi_k^1 - \xi_i \phi_k^2 + \epsilon_{ik} P_k) \det \Delta_{ik}^{(12)} = 0, \qquad i = 1, \ldots, N. \tag{86}$$

We recall that $\phi_k^{N-1} = r^k$, $\phi_k^N = 1$.

One can now introduce N matrices δ_i obtained from the matrix Δ by deleting the first two rows and the i-th column, and adding the first row with the elements $\phi_k^1 - \xi_i \phi_k^2 + \epsilon_{ik} P_k$. Thus, each matrix δ_i has dimension $(N-1) \times (N-1)$. Then the above set of equations (86) can be rewritten as

$$\det \delta_i = 0, \qquad i = 1, \ldots, N, \tag{87}$$

which implies that the rows of each of the matrices δ_i must be *linearly dependent*:

$$C_{i,1}\left(\epsilon_{ik} P_k + \phi_k^1 - \xi_i \phi_k^2\right) + \sum_{n=3}^{N-2} C_{i,n-1} \phi_k^n = C_{i,N-2} r^k + C_{i,N-1}, \qquad k = 1, \ldots N, \quad i = 1, \ldots, N-1\,, \quad k \neq i, \tag{88}$$

where $C_{i,k}$ are arbitrary constants. These conditions can be considered as N linear systems, for fixed k each. Since all these systems are consistent the functions ϕ_k^i and P_k can be found by solving system (88).

Constants $C_{i,1}$ cannot be equal to zero since in that case, according to (48), the velocities V^i would become undetermined. Therefore, without loss of generality we can set $C_{i,1} = 1$ and the number of free constants becomes $N(N-2)$. Thus, the following Proposition is valid:

Proposition 6.1: *General solution of system (86) is determined by solutions*

$$\phi_k^i = \frac{\det \tilde{B}_k^i r^k + \det \bar{B}_k^i}{\det B_k}, \qquad P_k = \frac{\det B_k^{(P)}}{\det B_k} \tag{89}$$

of N linear systems (88), where B_k, $\bar{B}_k^i$, $\tilde{B}_k^i$ and $B_k^{(P)}$ are matrices with elements

$$\tilde{b}_{kl}^{im} = \bar{b}_{kl}^{im} = b_{kl}^{(P)m} = b_{kl}^m = \begin{cases} 1 & \text{for } l=2 \\ -\xi_l & \text{for } l=3 \\ C_{m,l-2} & \text{for } l>3 \end{cases} \quad \text{if} \quad \begin{matrix} l \neq i+1 \\ l \neq 1 \end{matrix} \tag{90}$$

and $b_{k\,1}^{im} = \tilde{b}_{k\,1}^{im} = \bar{b}_{k\,1}^{im} = \epsilon_{mk}, \bar{b}_{ki}^{i+1\,m} = C_{i,N-1}$, $\tilde{b}_{ki}^{i+1\,m} = C_{i,N-2}, b_{k\,1}^{(P)m} = C_{i,N-2} r^k + C_{i,N-1}$,

where $C_{m,l}$ are arbitrary constants such that $\det B_k \neq 0$.

Remark: The set of constants $C_{l,m}$ for which $\det B_k = 0$ has Lebesque measure zero or requires a very special choice of the parameters η_k. The exceptional case is the following: the vectors ξ, $\mathbf{1}$ and ϵ_k are linearly dependent which yields, according to the definition (31), a set of equations for the special values η_k.

Thus, we have proved that all elements of the Stäckel matrix (47) depend linearly on Riemann invariants and these elements are determined from the algebraic system (30) up to $N(N-2)$ arbitrary constants removable by an appropriate change of the Riemann invariants (for instance, by a shift in the case $N=3$). By Corollary 4.1, the existence of such a Stäckel matrix automatically proves the semi-Hamiltonian and linearly-degenerate properties of the hydrodynamic reductions (29), (30).

Now, our main Theorem 3.1 is proved.

7 Riemann invariants and characteristic velocities: explicit construction

The construction described in Sections 3 and 6 provides a proof of the existence of Riemann invariants for system (29), (30) for arbitrary N. The Riemann invariants are found to parameterize system (29), (30) via the sole Stäckel matrix, which, by Corollary 4.1, implies linear degeneracy and integrability of this system. Explicit representations for conservation law densities u^i and transport velocities v^i in terms of the Riemann invariants are given by Ferapontov [9] formulae (49), (48) where the entries ϕ_k^n of the Stäckel matrix (47) and the functions $P_k(r^k)$ are defined by formulae (89) – (90). Using the functions ϕ_k^n one also obtains the generalised hodograph solutions (53).

The outlined procedure, while providing general theoretical framework for the study of the 'cold-gas' reductions of the kinetic equation for a soliton gas, seems to be not very convenient from the viewpoint of practical calculations. It also involves $N(N-2)$ intermediate constants $C_{l,m}$, which introduce an additional unnecessary complication. It is, thus, desirable to have more direct representations for the Riemann invariants and characteristic velocities, which will also be free from these intermediate arbitrary constants.

We shall make use of the Theorem 3.1 and show that, once the linear degeneracy and integrability properties of system (29), (30) are established, explicit relations between the

Riemann invariants $\mathbf{r}$ and the conserved densities $\mathbf{u}$ can be found by a straightforward calculation. The calculation will involve the properties of the Lamé coefficients outlined in Section 4.

First, without loss of generality we choose the following normalization (see (45))

$$u^k = \bar{H}_k, \tag{91}$$

where $\bar{H}_k$'s are the Lamé coefficients (42). Now, using Theorem 3.1 we assume that hydrodynamic type system (29), (30) can be rewritten in a diagonal form (36), so that $u^i = U^i(\mathbf{r})$ and $v^i(\mathbf{U}(\mathbf{r})) = V^i(\mathbf{r})$. For convenience, in what follows we shall use small u's and v's only, assuming that $u_j = u_j(\mathbf{r}) \equiv U^j(\mathbf{r})$, $v_j = v_j(\mathbf{r}) \equiv V_j(\mathbf{r})$.

To obtain explicit formulae for the Riemann invariants of the hydrodynamic reduction (29), (30) we need first to prove its so-called "Egorov" property.

Definition 7.1 (Pavlov & Tsarev 2003 [30]): *Semi-Hamiltonian hydrodynamic type system* (34) *is called the Egorov, if a* <u>*sole*</u> *pair of conservation laws*

$$\partial_t a(\mathbf{u}) = \partial_x b(\mathbf{u}), \quad \partial_t b(\mathbf{u}) = \partial_x c(\mathbf{u}) \tag{92}$$

exists. In this case (see (42) and (58)),

$$\partial_i a = \bar{H}_i^2, \quad \partial_i b = \tilde{H}_i \bar{H}_i, \quad \partial_i c = \tilde{H}_i^2, \tag{93}$$

while corresponding rotation coefficients (54) become symmetric, i.e.

$$\beta_{ik} = \beta_{ki}, \quad i \neq k.$$

Lemma 7.1: Hydrodynamic reductions (29), (30) are Egorov.

Proof: We consider the sum of conservation laws (29), (30)

$$\partial_t \left(\sum u^k\right) = \partial_x \left(\sum u^k v^k\right) = \partial_x \left[\sum_{k=1}^{N} u^k \left(\xi_k + \sum_{m\neq k} \epsilon_{km} u^m (v^m - v^k)\right)\right] \tag{94}$$

One can see that, since the matrix ϵ_{ik} is symmetric, the last term in r.h.s. of (94) vanishes. Thus, (94) simplifies to the form

$$\partial_t \left(\sum u^k\right) = \partial_x \left(\sum \xi_k u^k\right). \tag{95}$$

However, the *flux* $\Sigma \xi_k u^k$ of conservation law (95) is nothing but the *density* of another conservation law which can be obtained by the same summation but with the special weights ξ_i, i.e.

$$\partial_t \left(\sum \xi_k u^k\right) = \partial_x \left(\sum \xi_k u^k v^k\right).$$

Comparison with definition (92) implies that in our case

$$a = \sum u^m, \quad b = \sum \xi_m u^m \equiv \sum u^m v^m, \quad c = \sum \xi_m u^m v^m, \tag{96}$$

which completes the proof.

Now we formulate the following

Theorem 7.1: *The Riemann invariants of N-component hydrodynamic reductions (29), (30) can be found explicitly as*

$$r^i = -\frac{1}{u^i}\left(1+\sum_{m\neq i}\epsilon_{im}u^m\right), \qquad i=1,\dots,N. \tag{97}$$

Proof:

For the sake of completeness of our construction we first show that the linear degeneracy property (41) of system (29), (30) readily follows from the (assumed) existence of the Riemann invariants r^k. Indeed, differentiating (30) with respect to the Riemann invariant r^i and taking into account that (see (42), (91))

$$\partial_i \ln u^k = \frac{\partial_i v^k}{v^i - v^k}, \;\; i \neq k,$$

we obtain the expression

$$\partial_i v^i = \sum_{m\neq i}\epsilon_{im}(v^m - v^i)\partial_i u^m + \sum_{m\neq i}\epsilon_{im}u^m(\partial_i v^m - \partial_i v^i)\,,$$

which reduces, on using (44), to the form

$$\partial_i v^i\left(1+\sum_{m\neq i}\epsilon_{im}u^m\right) = 0\,. \tag{98}$$

Equation (98) can only be satisfied if $\partial_i v^i = 0$ for all i (otherwise the field variables u^m in the algebraic system (30) would cease to be independent). Thus system (29), (30) is indeed linearly degenerate.

Now, differentiation of algebraic system (30) with respect to the Riemann invariant r^k yields

$$\partial_k v^i = \sum_{m\neq i,k}\epsilon_{im}u^m(\partial_k v^m - \partial_k v^i) + \sum_{m\neq i,k}\epsilon_{im}(v^m - v^i)\partial_k u^m + \epsilon_{ik}u^k(\partial_k v^k - \partial_k v^i) + \epsilon_{ik}(v^k - v^i)\partial_k u^k,$$

which reduces, with an account of (44) and the linear degeneracy property, to

$$(v^k - v^i)\left[\left(1+\sum_{m\neq i}\epsilon_{im}u^m\right)\partial_k \ln u^i - \sum_{m\neq i}\epsilon_{im}\partial_k u^m\right] = 0.$$

Since all characteristic velocities v^k are distinct, the expression in square brackets must vanish for any pair of indices i and k, i.e. we have

$$\partial_k \ln u^i = \frac{\sum_{m\neq i}\epsilon_{im}\partial_k u^m}{\left(1+\sum_{m\neq i}\epsilon_{im}u^m\right)}, \quad k\neq i\,. \tag{99}$$

Integration of (99) yields

$$\sum_{m\neq i}\epsilon_{im}u^m + R_i(r^i)u^i = -1, \tag{100}$$

where $R_i(r^i)$, $i = 1,\dots,N$ are arbitrary functions.

We now differentiate (100) with respect to the Riemann invariants r^i and r^k, which gives, on using (91) and (54),

$$\sum_{m\neq i}\epsilon_{im}\beta_{im} + R_i'(r^i) + R_i(r^i)\partial_i \ln \bar{H}_i = 0 \tag{101}$$

and

$$\sum_{m\neq i,k}\epsilon_{im}\beta_{km} + R_i(r^i)\beta_{ki} + \epsilon_{ik}\partial_k \ln \bar{H}_k = 0 \tag{102}$$

respectively. Substitution of (96) into (93) gives

$$\bar{H}_i = \sum_{m\neq i}\beta_{im} + \partial_i \ln \bar{H}_i, \qquad \tilde{H}_i = \xi_i\bar{H}_i + \sum_{m\neq i}(\xi_m - \xi_i)\beta_{im}. \tag{103}$$

By expressing $\partial_i \ln \bar{H}_i$ from the above first equation, (101) and (102) reduce to the form

$$\begin{aligned} R_i(r^i)\bar{H}_i &= R_i(r^i)\sum_{m\neq i}\beta_{im} - \sum_{m\neq i}\epsilon_{im}\beta_{im} - R_i'(r^i), \\ \epsilon_{im}\bar{H}_m &= \epsilon_{im}\sum_{n\neq m}\beta_{nm} - \sum_{n\neq i,m}\epsilon_{in}\beta_{nm} - R_i(r^i)\beta_{im}. \end{aligned} \tag{104}$$

Substitution of the expressions $R_i(r^i)\bar{H}_i$ and $\epsilon_{im}\bar{H}_m$ into (100) yields to a set of constraints $R_i'(r^i) = 1$, i.e. $R_i(r^i) = r^i + \alpha_i$, where α_i are arbitrary constants. Since any function of the Riemann invariant is a Riemann invariant as well one can put without loss of generality that $R_i(r^i) = r^i$. Then (100) reduces to (97). The Theorem is proved.

Taking into account $R_i(r^i) = r^i$ and eliminating $\bar{H}_i$ from (104) we arrive at the linear algebraic system

$$\sum_{m\neq i,k}(r^i\epsilon_{km} - \epsilon_{ik}\epsilon_{im})\beta_{im} + (r^ir^k - \epsilon_{ik}^2)\beta_{ik} = \epsilon_{ik}, \quad i \neq k \tag{105}$$

for the rotation coefficients β_{ik}, while (104) reduces (cf. the first formula in (103)) to

$$\bar{H}_i = \sum_{m\neq i}\left(1 - \frac{\epsilon_{im}}{r^i}\right)\beta_{im} - \frac{1}{r^i}. \tag{106}$$

Now, using (58), (91), (103) and (106) we formulate the main result of this Section:

Algebraic relations (30) *can be resolved in a parametric form:*

$$u^i = \sum_{m\neq i}\left(1 - \frac{\epsilon_{im}}{r^i}\right)\beta_{im} - \frac{1}{r^i}, \qquad v^i = \xi_i + \sum_{m\neq i}(\xi_m - \xi_i)\frac{\beta_{im}}{u^i}, \tag{107}$$

where the symmetric rotation coefficients β_{ik} are found from the linear system (see (105)),

$$\sum_{m\neq i,k}(r^i\epsilon_{km}-\epsilon_{ik}\epsilon_{im})\beta_{im}+(r^ir^k-\epsilon_{ik}^2)\beta_{ik}=\epsilon_{ik},\quad i\neq k. \tag{108}$$

Thus, we have obtained unambiguous expressions for the Riemann invariants and characteristic velocities of the N-component hydrodynamic reductions (29), (30).

In particular, for $N=3$ we have from (107), (108) explicit expressions for the rotation coefficients and characteristic velocities

$$\begin{aligned}
\beta_{12} &= \frac{r^3\epsilon_{12}-\epsilon_{13}\epsilon_{23}}{r^1r^2r^3-r^1\epsilon_{23}^2-r^2\epsilon_{13}^2-r^3\epsilon_{12}^2+2\epsilon_{12}\epsilon_{13}\epsilon_{23}},\\
\beta_{13} &= \frac{r^2\epsilon_{13}-\epsilon_{12}\epsilon_{23}}{r^1r^2r^3-r^1\epsilon_{23}^2-r^2\epsilon_{13}^2-r^3\epsilon_{12}^2+2\epsilon_{12}\epsilon_{13}\epsilon_{23}},\\
\beta_{23} &= \frac{r^1\epsilon_{23}-\epsilon_{12}\epsilon_{13}}{r^1r^2r^3-r^1\epsilon_{23}^2-r^2\epsilon_{13}^2-r^3\epsilon_{12}^2+2\epsilon_{12}\epsilon_{13}\epsilon_{23}};
\end{aligned} \tag{109}$$

$$\begin{aligned}
v^1 &= \frac{\xi_1\,(\epsilon_{23}^2-r^2r^3)+\xi_2\,(r^3\epsilon_{12}-\epsilon_{13}\epsilon_{23})+\xi_3\,(r^2\epsilon_{13}-\epsilon_{12}\epsilon_{23})}{\epsilon_{23}^2-r^2r^3+\epsilon_{13}\,(r^2-\epsilon_{23})+\epsilon_{12}\,(r^3-\epsilon_{23})}\\
v^2 &= \frac{\xi_2\,(\epsilon_{13}^2-r^1r^3)+\xi_3\,(r^1\epsilon_{23}-\epsilon_{12}\epsilon_{13})+\xi_1\,(r^3\epsilon_{12}-\epsilon_{13}\epsilon_{23})}{\epsilon_{13}^2-r^1r^3+\epsilon_{13}(r^3-\epsilon_{13})+\epsilon_{23}(r^1-\epsilon_{13})}\\
v^3 &= \frac{\xi_3\,(\epsilon_{12}^2-r^1r^2)+\xi_2\,(r^1\epsilon_{23}-\epsilon_{12}\epsilon_{13})+\xi_1\,(r^2\epsilon_{13}-\epsilon_{12}\epsilon_{23})}{\epsilon_{12}^2-r^1r^2+\epsilon_{13}(r^2-\epsilon_{12})+\epsilon_{23}(r^1-\epsilon_{12})}
\end{aligned} \tag{110}$$

One can observe that formulae (110) do not coincide with representation (70), (68) obtained earlier for the same family of the characteristic velocities. The reason is that the two representations correspond to different choices of the Riemann invariants (we recall one more time that any function of a Riemann invariant is a Riemann invariant as well). The relationship between these two equivalent sets of the Riemann invariants is obtained by equating the characteristic velocities (70) and (110). As a result we get

$$\begin{aligned}
\tilde r^1 &= \frac{(\epsilon_{13}-\epsilon_{12})(\epsilon_{23}r^1-\epsilon_{12}\epsilon_{13})}{(\xi_2-\xi_3)r^1+(\xi_3-\xi_1)\epsilon_{12}+(\xi_1-\xi_2)\epsilon_{13}},\\
\tilde r^2 &= \frac{(\epsilon_{12}-\epsilon_{23})(\epsilon_{13}r^2-\epsilon_{12}\epsilon_{23})}{(\xi_3-\xi_1)r^2+(\xi_1-\xi_2)\epsilon_{23}+(\xi_2-\xi_3)\epsilon_{12}},\\
\tilde r^3 &= \frac{(\epsilon_{23}-\epsilon_{13})(\epsilon_{12}r^3-\epsilon_{13}\epsilon_{23})}{(\xi_1-\xi_2)r^3+(\xi_2-\xi_3)\epsilon_{13}+(\xi_3-\xi_1)\epsilon_{23}}.
\end{aligned} \tag{111}$$

Here by $\tilde r^1$, $\tilde r^2$, $\tilde r^3$ we denote the 'old' Riemann invariants as in (70).

8 Commuting hydrodynamic flows

8.1 General explicit representation

Commuting flows to semi-Hamiltonian linearly degenerate system (29), (30) are defined in terms of the Riemann invariants by equations (40), (39). We recall that, according to

Proposition 4.1, only $N-2$ of the commuting flows are linearly degenerate (excluding the 'trivial' flows specified by linear combinations of the constant characteristic velocity $\mathbf{1}$ and the characteristic velocity $\mathbf{v}$ of the original flow (36)). The general solution of the generalised hodograph equations (39) specifying commuting flows was obtained by Ferapontov [9] in terms of the Stäckel matrix entries (see Theorem 4.2). Here we are interested in a more explicit representation of the commuting flows for the specific system (29), (30). For that, instead of integrating system (39), we take advantage of the fact that our linearly degenerate system (29), (30) is Egorov. In that case, the commuting flows can be found explicitly.

We first observe that any conservation law density h for linearly degenerate hydrodynamic type system (29) can be represented in the form (see (45) or (49))

$$h=\sum_{k=1}^{N}u^kP_k(r^k), \tag{112}$$

with N arbitrary functions $P_k(r^k)$ of a single variable. Then we make use of

Lemma 8.1 (Pavlov & Tsarev 2003 [30]): *All commuting flows (40), (39) in the Egorov case are specified by the expression (see (57), (93))*

$$W^i=\frac{H_i}{\bar{H}_i}=\frac{\partial_i h}{\partial_i a}. \tag{113}$$

Substituting (112), (96) into (113) and using the first formula from (103) we obtain an explicit representation for the characteristic velocities of the commuting flows (40),

$$W^i=P_i(r^i)+\frac{1}{\bar{H}_i}\left(P_i'(r^i)+\sum_{m\neq i}(P_m(r^m)-P_i(r^i))\beta_{im}\right). \tag{114}$$

We recall that $P_k(r^k)$, $k=1,\dots,N$ are arbitrary functions and the dependence of the rotation coefficients β_{im} on the Riemann invariants is found from system (108).

If $P_k(r^k)=1$, (114) reduces to $W^i=1$; if $P_k(r^k)=\xi_k$, it reduces to the second formula in (107), i.e. to hydrodynamic reduction (29), (30) itself.

8.2 Linearly degenerate commuting flows

To extract the family of linearly degenerate commuting flows from general representation (114) we formulate the following

Lemma 8.2: *For the linearly degenerate commuting flows each function $P_i(r^i)$ in (114) is linear with respect to the corresponding Riemann invariant r^i.*

Proof: The condition $\partial_i W^i=0$ of linear degeneracy of the commuting flow implies, on using (59) and (103), that $P_i''(r^i)=0$.

We now consider the representation for the family of linearly degenerate commuting flows suggested by the form of the kinetic equations (26), (27) for the KdV hierarchy. Importantly, the whole KdV kinetic hierarchy (26), (27) is characterised by a single integral kernel, $G(\eta,\mu)=\ln|(\eta-\mu)/(\eta+\mu)|$ (which is consistent with the fact that all equations of the original finite-gap Whitham hierarchy are associated with the same Riemann surface).

This suggests that there could exist a family of commuting flows to general nonlocal kinetic equation (1) having the form

$$f_\tau = (\tilde{s} f)_x \,,$$
$$\tilde{s}(\eta) = \tilde{S}(\eta) + \frac{1}{\eta}\int\limits_0^\infty G(\eta,\mu) f(\mu)[\tilde{s}(\mu) - \tilde{s}(\eta)]d\mu \,, \tag{115}$$

where $\tilde{S}(\eta)$ is an arbitrary function. Although verification of commutativity of the kinetic equations (1) and (115) is beyond the scope of the present paper, it is clear that, if these equation do commute, this must be manifested on the level of hydrodynamic reductions as well. Having this in mind, we consider the N-component hydrodynamic reductions to (115) obtained by the familiar delta-functional ansatz (28) and try to see if they commute with the original reductions (29)–(31).

First we notice that equation (115) is, essentially, the same kinetic equation (1) but with a different time variable and different "free soliton speed" function $S(\eta)$. Now, since we have proved integrability of the linearly degenerate hydrodynamic reductions (29)–(31) in a general form, we automatically have that analogous N-component hydrodynamic reductions of (115) must also be integrable linearly degenerate systems. It should be noted that, since the function $\tilde{S}(\eta)$ is arbitrary, the set $\{\tilde{\xi}_1, \dots, \tilde{\xi}_N\}$ of its values $\tilde{\xi}_j = \tilde{S}(\eta_j)$ can be viewed as a set of N arbitrary numbers, and the corresponding 'cold-gas' hydrodynamic reduction becomes (cf. (29)–(31))

$$u^i_\tau = (u^i \tilde{v}^i)_x, \qquad i = 1, \dots, N, \tag{116}$$

where the velocities $\tilde{v}^i = \tilde{s}^i$ and the conservation law densties u^i satisfy algebraic relations

$$\tilde{v}^i = \tilde{\xi}_i + \sum_{k=1}^{N} \epsilon_{ik} u^k (\tilde{v}^k - \tilde{v}^i), \qquad \epsilon_{ik} = \epsilon_{ki}, \qquad \epsilon_{ii} = 0, \tag{117}$$

and ϵ_{ik} are the same as in (31).

According to Theorem 3.1, equations (116), (117) can be represented in the Riemann form

$$r^i_\tau = \tilde{v}^i(\mathbf{r}) r^i_x, \quad i = 1, 2, ..., N; \quad k = 1, 2, ... \tag{118}$$

where the dependence $\tilde{v}^i(\mathbf{r})$ of the characteristic velocities on the Riemann invariants is determined by the same formulae (107) with the only difference that, one now replaces ξ_j with $\tilde{\xi}_j$, i.e. we have

$$\tilde{v}^i = \tilde{\xi}_i + \sum_{m \neq i} (\tilde{\xi}_m - \tilde{\xi}_i) \frac{\beta_{im}}{u^i}. \tag{119}$$

Indeed, representation (119) is a straightforward consequence of (107) since the rotation coefficients β_{im} and Lamé coefficients $\bar{H}_i$ do not depend on the parameters ξ_j (see (108), first formula in (107), and normalisation (91)). It not difficult to see that commutativity relationships (see (39))

$$\frac{\partial_i \tilde{v}^j}{\tilde{v}^i - \tilde{v}^j} = \frac{\partial_i v^j}{v^i - v^j}\,, \qquad i, j = 1, 2, 3, \quad i \neq j\,, \tag{120}$$

are identically satisfied. Thus, we have proved the following

Lemma 8.3: *Linearly degenerate semi-Hamiltonian flows (116), (117) and (29), (30) commute for any N.*

In conclusion we note that, although we have proved integrability of the 'cold-gas' hydrodynamic reductions (29), (30) for an arbitrary choice of the functions $S(\eta)$ and $G(\eta, \mu)$ in the original kinetic equation (1), one can expect that integrability of the full equation (1) would require some additional restrictions on the integral kernel $G(\eta, \mu)$ (other than just symmetry).

Acknowledgments

We are grateful to V.E. Zakharov for his interest in this work and for clarifying the place of the kinetic equation for solitons within the general theory of "integrable turbulence". We thank Yu. Fedorov, E. Ferapontov, O. Morozov, A. Neishtadt, S. Tsarev and A. Veselov for stimulating discussions. The work has been partially supported by EPSRC (UK)(grant EP/E040160/1) and London Mathematical Society (Scheme 4 Collaborative Visits Grant). Work of M.V.P. has been also supported by the Programme "Fundamental problems of nonlinear dynamics" of Presidium of RAS. M.V.P. and S.A.Z. also acknowledge partial financial support from the Russian–Taiwanese grant 95WFE0300007 (RFBR grant 06-01-89507-HHC) .

References

[1] Arik, M., Neyzi, F., Nutku, Ya. & Olver, P., Multi-Hamiltonian structure of the Born–Infeld equation, *J. Math. Phys.* **30** (1989) 1338-1344 .

[2] Belokolos, E.D., Kinetic equations and integrable Hamiltonian systems, *Ukr. Math. Journ.* **57** (2005) 869 – 882.

[3] Benney, D.J., Some properties of long nonlinear waves, *Stud. Appl. Math.* **52** (1973) 45-50.

[4] Born, M. & Infeld, L., Foundations of a new field theory *Proc. Roy. Soc. A* **144** (1934) 425-451.

[5] El, G.A., The thermodynamic limit of the Whitham equations, *Phys. Lett. A* **311**(2003) 374 - 383.

[6] El, G.A., Krylov, A.L., Molchanov, S.A. & Venakides, S., Soliton turbulence as the thermodynamic limit of stochastic soliton lattices, *Physica D* **152-153** (1999) 653

[7] El, G.A. & Kamchatnov, A.M., Kinetic equation for a dense soliton gas, *Phys. Rev. Lett.* **95** (2005) Art. No 204101.

[8] G. Darboux, Leçons sur les systèmes orthogonaux et les coordonnées curvilignes, Paris (1910).

[9] Ferapontov, E.V., Integration of weakly-nolinear hydrodynamic systems in Riemann invariants, *Phys. Lett. A* **158** (1991) 112-118.

[10] Ferapontov, E.V. & Khusnutdinova, K.R., On integrability of (2+1)-dimensional quasilinear systems, *Comm. Math. Phys.* **248** (2004) 187-206,

[11] Ferapontov, E.V. and Khusnutdinova, K.R., The characterization of 2-component (2+1)-dimensional integrable systems of hydrodynamic type, *J. Phys. A: Math. Gen.* **37** (2004) 2949 - 2963.

[12] Ferapontov, E.V. and Marshall, D.G., Differential-geometric approach to the integrability of hydrodynamic chains: the Haantjes tensor, *Matematische Annalen* **339** (2007) 61-99.

[13] Flaschka, H., Forest, G., McLaughlin, D.W., Multiphase averaging and the inverse spectral solutions of the Korteweg – de Vries equation. *Comm. Pure Appl. Math.* **33** (1980) 739-784 .

[14] Gantmaher, F.R., *Applications of the theory of matrices*, Interscience, New York (1959).

[15] Gibbons, J., Collisionless Boltzmann equations and integrable moment equations, *Physica D* **3** (1981) 503-511.

[16] Gibbons, J. & Tsarev, S.P., Reductions of Benney's equations, *Phys. Lett. A*, **211** (1996) 19-24.

[17] Gibbons, J. & Tsarev, S.P., Conformal maps and reductions of the Benney equations, *Phys. Lett. A*, **258** (1999) 263-270.

[18] Gibbons, J. & Raimondo, A., Differential geometry of hydrodynamic Vlasov equations, *Journ. Geom. Phys.* **57** (2007) 1815-1828.

[19] Gibbons, J. & Kodama, Yu., Solving dispersionless Lax equations. In N. Ercolani et al., editor, *Singular limits of dispersive waves*, v. **320** of NATO ASI Series B, page 61. Plenum, New York (1994).

[20] Gurevich, A.V., Mazur, N.G., and Zybin, K.P., Statistical limit in a completely integrable system with deterministic initial conditions, *Journ. Exp. Theor. Phys.* **90** (2000) 797 - 817.

[21] Johnson, R and Moser, J., The rotation number for almost periodic potentials *Comm. Math. Phys.* **84** (1982) 403-438

[22] Kodama, Yu., A method for solving the dispersionless KP equation and its exact solutions. *Phys. Lett. A* **129** (1988) 223-226.

[23] Kodama, Yu., A solution method for the dispersionless KP equation, *Prog. Theor. Phys. Supplement.* **94** (1988) 184.

[24] Kotani, S., KdV flow on generalized reflectionless potentials, *Journ of Math. Phys. Anal. Geom* **4** (2008) 490 - 528

[25] Lax, P.D., The zero dispersion limit, a deterministic analog of turbulence, *Comm. Pure Appl. Math.* **44** (1991) 1047-1056.

[26] Novikov, S.P., Manakov, S.V., Pitaevskii, L.P. and Zakharov, V.E., *The Theory of Solitons: The Inverse Scattering Method*. Consultants, New York (1984).

[27] Odesskii, A.V., Pavlov, M.V., Sokolov, V.V. Classification of integrable Vlasov-like equations, *Theor. Math. Phys.* **154** (2008) 248-259.

[28] Pavlov, M.V., Hamiltonian formalism of weakly nonlinear systems in hydrodynamics *Theor. Math. Phys.* **73** (1987) 1242-1245.

[29] Pavlov, M.V., Algebro-geometric approach in the theory of integrable hydrodynamic type systems. *Comm. Math. Phys.* **272** (2007) 469-505.

[30] Pavlov, M.V., Tsarev, S.P., Three-Hamiltonian structures of the Egorov hydrodynamic type systems, *Funct. Anal. Appl.*, **37**, No. 1 (2003) 32-45.

[31] Tsarev, S.P., On Poisson brackets and one-dimensional Hamiltonian systems of hydrodynamic type, *Soviet Math. Dokl.* **31** (1985) 488–491.

[32] Tsarev, S.P., The geometry of Hamiltonian systems of hydrodynamic type. The generalized hodograph method, *Math. USSR Izvestiya* ***37*** (1991) 397–419.

[33] Whitham, G.B., *Linear and Nonlinear Waves* Wiley, New York (1974).

[34] Venakides, S., The continnum limit of theta functions, *Comm.Pure Appl.Math.* **42** (1989) 711.

[35] Zakharov, V.E., Kinetic equation for solitons, *Sov. Phys. JETP* **33** (1971) 538-541

[36] Zakharov, V.E., On the Benney equations, *Physica D* **3** (1981) 193-202.

[37] Zakharov, V.E., Dispersionless limit of integrable systems in 2+1 dimensions, in *Singular Limits of Dispersive Waves*, Ed. N.M. Ercolani et al., Plenum Press, NY (1994) 165-174.

[38] Zakharov, V.E. Turbulence in integrable systems *Stud. Appl. Math.* **122** (2009) 219 -234.