

Almost sure Nash equilibrium strategies in evolutionary models of asset markets

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Abstract

We consider a stochastic model of a financial market with long-lived dividend-paying assets and endogenous asset prices. The model was initially developed and analyzed in the context of evolutionary finance, with the main focus on questions of “survival and extinction” of investment strategies. In this paper we view the model from a different, game-theoretic, perspective and examine Nash equilibrium strategies, satisfying equilibrium conditions with probability one.

Key words: stochastic games, evolutionary finance, capital growth theory, random dynamical systems

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1 Introduction

We examine a stochastic model of a financial market with long-lived dividend-paying assets and endogenous market clearing asset prices. The model is a version of that proposed in Evstigneev et al. (2006) and then analyzed primarily in the context of evolutionary finance (for a survey of the field see Evstigneev et al. (2009)). The main focus in evolutionary finance is on questions of “survival and extinction” of investment strategies (portfolio rules). In this paper we view the model from a different perspective and treat its decision-theoretic framework as a game in which the payoffs of the players (investors) are defined in terms of the growth rates of their relative wealth. We show that in the game under consideration the Kelly (1956) portfolio rule of “betting your beliefs” forms with probability one a unique symmetric Nash equilibrium strategy.

Game-theoretic models of asset markets dealing with relative wealth of investors were considered by Bell and Cover (1980, 1988). In those models, the objectives of the players were described in terms of the expectations of random payoff functions. The notion of a Nash equilibrium defined in terms of these expectations was used as a game solution concept. We consider a different (stronger) solution concept : almost sure Nash equilibrium. According to our definition of an equilibrium strategy, any unilateral deviation from it leads to a decrease in the random payoff with probability one, and not only to a decrease in the expected payoff.

The focus on optimality almost surely is characteristic for capital growth theory (Kelly (1956), Breiman (1961), Algoet and Cover (1988), Hakansson and Ziemba (1995), MacLean et al. (2010)), dealing in its classical version with exogenous asset prices. Results related to evolutionary finance models may be regarded as analogues, and in certain cases as generalizations, of those pertaining to classical models of capital growth. A fundamental distinction between the two modelling frameworks lies in the fact that in the former the dynamics of wealth of an investor might depend (via the endogenous price formation mechanism) not only on his/her strategy, but also on the strategies used by the other investors. Therefore in the present context a game-theoretic model, rather than a single-agent optimization framework, is a suitable setting for the analysis of questions related to capital growth.

The paper is organized as follows. Section 2 describes the model, Section 3 states the main results and Section 4 gives the proof of the main theorem.

The Appendix contains the proof of a technical lemma.

2 The model

We consider a market where $K \geq 2$ assets are traded. The market is influenced by random factors modeled in terms of independent identically distributed random elements s_1, s_2, \dots in a measurable space S . At each date $t = 1, 2, \dots$ assets $k = 1, 2, \dots, K$ pay dividends $D_k(s_t) \geq 0$ depending on the “state of the world” s_t at date t . The functions $D_k(s_t)$ are measurable and satisfy

$$\sum_{k=1}^K D_k(s) > 0 \text{ for all } s. \quad (1)$$

This condition means that in each random situation at least one asset yields a strictly positive dividend. The total volume (the number of units) of asset k traded in the market at date t is $V_{t,k} = V_{t,k}(s^t) > 0$, where $s^t := (s_1, \dots, s_t)$ is the history of the process (s_t) from time 1 to time t . For $t = 0$, $V_{t,k}$ is a constant number, and for $t \geq 1$, $V_{t,k}(s^t)$ is a measurable function of s^t .

We denote by $p_t \in \mathbb{R}_+^K$ the vector of market prices of the assets. For each $k = 1, \dots, K$, the coordinate $p_{t,k}$ of $p_t = (p_{t,1}, \dots, p_{t,K})$ stands for the price of one unit of asset k at date t . There are $N \geq 2$ investors (traders) acting in the market. A portfolio of investor i at date $t = 0, 1, \dots$ is specified by a vector $x_t^i = (x_{t,1}^i, \dots, x_{t,K}^i) \in \mathbb{R}_+^K$ where $x_{t,k}^i$ is the amount (the number of units) of asset k in the portfolio x_t^i . The scalar product $\langle p_t, x_t^i \rangle = \sum_{k=1}^K p_{t,k} x_{t,k}^i$ expresses the value of the investor i 's portfolio x_t^i at date t in terms of the prices $p_{t,k}$.

At date $t = 0$ the investors have initial endowments $w_0^i > 0$ ($i = 1, 2, \dots, N$) that form their budgets at date 0. Investor i 's budget (wealth) at date $t \geq 1$ is

$$w_t^i := \langle D_t + p_t, x_{t-1}^i \rangle, \quad (2)$$

where

$$D_t := D(s_t) := (D_1(s_t), \dots, D_K(s_t)).$$

It consists of two components: the dividends $\langle D_t, x_{t-1}^i \rangle$ paid by the portfolio x_{t-1}^i and the market value $\langle p_t, x_{t-1}^i \rangle$ of the portfolio x_{t-1}^i expressed in terms of the today's prices p_t . A fraction $\alpha_t = \alpha_t(s^{t-1})$ of the budget is invested into assets. We suppose that the investment rate $0 < \alpha_t(s^{t-1}) < 1$ is the same for all the traders, although it may depend on time and random factors. We assume that α_t is predictable: it depends on the history s^{t-1} of the process (s_t) up to time $t - 1$ (not t). The number $1 - \alpha_t$ can represent the tax

rate or the *consumption rate*. The assumption that $1 - \alpha_t$ is the same for all the investors is quite natural in the former case. In the latter case it is indispensable since we focus in this work on the analysis of the comparative performance of trading strategies (portfolio rules) in the long run. Without this assumption, an analysis of this kind does not make sense: a seemingly worse performance of a portfolio rule in the long run might be simply due to a higher consumption rate of the investor.

We shall suppose that the function $\alpha_t(s^{t-1})$ is measurable (for $t = 0, 1$ it is constant) and satisfies the following condition:

$$\alpha_t(s^{t-1}) < V_{t,k}(s^t)/V_{t-1,k}(s^{t-1}). \quad (3)$$

This condition holds, in particular, when the total mass $V_{t,k}(s^t)$ of each asset k does not decrease, i.e. when the right-hand side of (3) is not less than one. But (3) does not exclude the situation when $V_{t,k}$ decreases at some rate, not faster than α_t .

An *investment strategy* (*portfolio rule*) of trader $i = 1, 2, \dots, N$ is specified by a vector of *investment proportions* $\lambda_t^i = (\lambda_1^i, \dots, \lambda_K^i)$ according to which he/she plans to distribute the available budget between assets at each date t . Vectors λ^i belong to the unit simplex

$$\Delta^K := \{(a_1, \dots, a_K) \geq 0 : a_1 + \dots + a_K = 1\}.$$

Strategies of this kind are called *fixed-mix*, or *constant proportions*, portfolio rules: they prescribe to select investment proportions at time 0 and keep them fixed over the whole infinite time horizon.

The class of fixed-mix strategies we consider in this work is quite common in financial theory and practice; see, e.g., Perold and Sharpe (1988) and Browne (1998). Under certain conditions, strategies with constant investment proportions lead to the growth of the portfolio value (“volatility pumping”—Luenberger (1998)). From the theoretical standpoint, this class of portfolio rules provides a convenient laboratory for the analysis of questions we are interested in. It makes it possible to formalize in a clear and compact way the concept of the *type* of an investor which determines the performance of his/her portfolio rule in the long run. A similar approach is common in evolutionary game theory (e.g. Weibull (1995)).

In the model at hand, the asset market evolves in time, remaining in the state of a dynamic *short-run (temporary) equilibrium*¹. The notion of market

¹In this paper, we use the term “equilibrium” in two different meanings. Here it is related to market equilibrium: a situation when asset supply is equal to asset demand. Later, the same term will appear in a game-theoretic context. It will pertain to Nash equilibrium strategies in a certain dynamic game.

equilibrium is defined as follows. Suppose each investor $i = 1, \dots, N$ has selected a strategy – a vector of investment proportions $\lambda^i = (\lambda_1^i, \dots, \lambda_K^i) \in \Delta^K$. Then at date $t \geq 0$, the amount invested in asset k by trader i is $\alpha_t \lambda_k^i w_t^i$ and the total amount invested in asset k is $\alpha_t \sum_{i=1}^N \lambda_k^i w_t^i$, where w_t^i is the i 's budget. It is assumed that the market clears (asset supply is equal to asset demand), which makes it possible to determine the equilibrium price $p_{t,k}$ of each asset k from the equations

$$p_{t,k} V_{t,k} = \alpha_t \sum_{i=1}^N \lambda_k^i w_t^i, \quad k = 1, \dots, K. \quad (4)$$

On the left-hand side of (4) we have the total value $p_{t,k} V_{t,k}$ of all the assets of the type k in the market (recall that the amount of each asset k at date t is $V_{t,k}$). The right-hand side represents the total wealth invested in asset k by all the investors. Equilibrium implies the equality in (4). The investment proportions $\lambda_1^i, \dots, \lambda_K^i$ chosen by the traders determine their portfolios $x_t^i = (x_{t,1}^i, \dots, x_{t,K}^i)$ at date t by the formula

$$x_{t,k}^i = \frac{\alpha_t \lambda_k^i w_t^i}{p_{t,k}}, \quad k = 1, \dots, K, \quad i = 1, \dots, N. \quad (5)$$

Note that for $t \geq 1$, the price vector p_t is determined implicitly as the solution to the system of equations (4), which can be written

$$p_{t,k} V_{t,k} = \alpha_t \sum_{i=1}^N \lambda_k^i \langle D_t + p_t, x_{t-1}^i \rangle, \quad k = 1, \dots, K. \quad (6)$$

It can be shown that under assumption (3) a non-negative vector p_t satisfying these equations exists and is unique (for any s^t and any feasible x_{t-1}^i and λ^i)—see Proposition 1 in Amir et al. (2009).

Given a strategy profile $(\lambda^1, \dots, \lambda^N)$ of investors and their initial endowments w_0^1, \dots, w_0^N , we can generate a path

$$(p_t; x_t^1, \dots, x_t^N), \quad (7)$$

of market dynamics, by defining the price vectors $p_t = p_t(s^t)$ and the portfolios $x_t^i = x_t^i(s^t)$ recursively according to equations (4)–(5).

A comment regarding the above description of asset market dynamics is in order. Equations (5) make sense only if $p_{t,k} > 0$, or equivalently, if the aggregate demand for each asset (under the equilibrium prices) is strictly positive. Those strategy profiles $(\lambda^1, \dots, \lambda^N)$ which guarantee that the recursive procedure described above leads at each step to strictly positive equilibrium prices will be called *admissible*. In what follows, we will deal only

with such strategy profiles. The hypothesis of admissibility guarantees that the random dynamical system under consideration is well-defined. Under this hypothesis, we obtain by induction that on the equilibrium path, all the portfolios $x_t^i = (x_{t,1}^i, \dots, x_{t,K}^i)$ are non-zero and the wealth $w_t^i = \langle D_t + p_t, x_{t-1}^i \rangle$ of each investor is strictly positive. Further, by summing up equations (5) over $i = 1, \dots, N$, we find that

$$\sum_{i=1}^N x_{t,k}^i = \frac{\sum_{i=1}^N \alpha_t \lambda_k^i w_t^i}{p_{t,k}} = \frac{p_{t,k} V_{t,k}}{p_{t,k}} = V_{t,k} \quad (8)$$

(the market clears) for every asset k and each date $t \geq 1$. Thus for every equilibrium state of the market $(p_t, x_t^1, \dots, x_t^N)$, we have $p_t > 0$, $x_t^i \neq 0$ and (8).

We give a simple sufficient condition for a strategy profile to be admissible which will hold for all the strategy profiles we shall deal with in the present paper. Suppose that some trader, say trader 1, uses a portfolio rule that prescribes to invest into all the assets in strictly positive proportions λ_k^1 (a *completely mixed* portfolio rule). Then a strategy profile containing this portfolio rule is admissible. Indeed, for $t = 0$, we get from (4) that $p_{0,k} \geq \alpha_0 V_{0,k}^{-1} \lambda_k^1 w_0^1 > 0$ and from (5) that $x_0^1 = (x_{0,1}^1, \dots, x_{0,K}^1) > 0$ (coordinatewise). Assuming that $x_{t-1}^1 > 0$ and arguing by induction, we obtain $\langle D_t + p_t, x_{t-1}^1 \rangle \geq \langle D_t, x_{t-1}^1 \rangle > 0$ in view of (1), which in turn yields $p_t > 0$ and $x_t^1 > 0$ by virtue of (4) and (5), as long as $\lambda_k^1 > 0$.

3 The main results

Let $(\lambda^1, \dots, \lambda^N)$ be an admissible strategy profile of the investors. Consider the path (7) of the random dynamical system generated by this strategy profile and the given initial endowments. As above, let w_t^i denote the investor i 's wealth available at date $t \geq 0$. If $t = 0$, then w_0^i is a constant number, the initial endowment of investor i . If $t \geq 1$, then $w_t^i = w_t^i(s^t)$ is a measurable function of s^t given by formula (2). As we have noted above, $w_t^i(s^t) > 0$.

We are primarily interested in the long-run behavior of the *relative wealth* of the investors (players). Given a strategy profile $(\lambda^1, \dots, \lambda^N)$, the performance of a strategy λ^i used by investor i will be characterized by the following random variable

$$\xi^i := \limsup_{t \rightarrow \infty} \frac{1}{t} \ln \frac{w_t^i}{\sum_{j \neq i} w_t^j}.$$

The expression $w_t^i / \sum_{j \neq i} w_t^j$ is the relative wealth of player i and the group $\{j : j \neq i\}$ of i 's rivals. The random variable $\xi^i = \xi^i(s^\infty; \lambda^1, \dots, \lambda^N)$ depends

on the strategy profile $(\lambda^1, \dots, \lambda^N)$ and on the whole history $s^\infty := (s_1, s_2, \dots)$ of states of the world from time 1 to ∞ . In the game under consideration, ξ^i plays the role of the (random) *payoff function* of player i .

We shall say that a strategy $\bar{\lambda}$ forms a *symmetric Nash equilibrium almost surely (a.s.)* if

$$\xi^i(s^\infty; \bar{\lambda}, \dots, \bar{\lambda}) \geq \xi^i(s^\infty; \bar{\lambda}, \dots, \lambda, \dots, \bar{\lambda}) \text{ (a.s.)} \quad (9)$$

for every i , each strategy λ of investor i and each set of initial endowments $w_0^1 > 0, \dots, w_0^N > 0$. The Nash equilibrium is called *strict* if the inequality in (9) is strict.

Recall that we consider only those strategy profiles which are admissible. If all the players use the same strategy $\bar{\lambda}$, then the strategy profile $(\bar{\lambda}, \dots, \bar{\lambda})$ is admissible if and only if the vector $\bar{\lambda}$ is strictly positive. This is immediate from (6).

Assume that the total mass $V_{t,k}$ of each asset k grows (or decreases) at the same rate $\gamma_t = \gamma_t(s^{t-1}) > 0$:

$$V_{t,k}/V_{t-1,k} = \gamma_t \text{ (} t \geq 1 \text{)}. \quad (10)$$

Thus

$$V_{t,k}(s^{t-1}) = \gamma_t(s^{t-1}) \dots \gamma_2(s_1) \gamma_1 V_k, \quad (11)$$

where $V_k > 0$ ($k = 1, 2, \dots, K$) are the initial amounts of the assets. The growth rate process γ_t (like the investment rate process α_t) is predictable: γ_t depends only on the history s^{t-1} of the states of the world up to time $t-1$. In the case of dividend-paying assets involving investments in the real economy, assumption (10) means that the economic system under consideration is on a *balanced growth path*.

Define the *relative dividends* of the assets $k = 1, \dots, K$ by

$$R_k(s_t) = \frac{D_k(s_t)V_k}{\sum_{m=1}^K D_m(s_t)V_m} \quad (12)$$

It follows from (10) that

$$R_{t,k} = \frac{D_{t,k}V_{t-1,k}}{\sum_{m=1}^K D_{t,m}V_{t-1,m}}.$$

where $R_{t,k} = R_k(s_t)$ and $D_{t,k} = D_k(s_t)$. Define

$$\lambda_k^* = ER_k(s_t), \quad k = 1, 2, \dots, K \quad (13)$$

and put $\lambda^* = (\lambda_1^*, \dots, \lambda_K^*)$. The portfolio rule specified by (13) prescribes to distribute wealth across assets in accordance with the proportions of the

expected relative dividends (which do not depend on t because the random elements s_t are i.i.d.).

The strategy λ^* is a generalization of the Kelly portfolio rule of “betting your beliefs” playing an important role in capital growth theory—see Kelly (1956) and MacLean et al. (2010). The “beliefs” are expressed here in terms of the expected relative dividends.

Assume that the following conditions hold.

(R1) For each k , the expectation $ER_k(s_t)$ is strictly positive.

(R2) The functions $R_1(s), \dots, R_K(s)$ are linearly independent with respect to the probability distribution of s_t , i.e. the equality $\sum \gamma_k R_k(s_t) = 0$ holding a.s. for some constants γ_k implies that $\gamma_1 = \dots = \gamma_K = 0$.

(R3) There exist constants $0 < \rho' < \rho'' < 1$ such that the process

$$\rho_t(s^{t-1}) := \alpha_t(s^{t-1})/\gamma_t(s^{t-1})$$

satisfies $\rho' \leq \rho_t(s^{t-1}) \leq \rho''$.

Condition (R1) implies that the vector λ^* has strictly positive coordinates. Hypothesis (R2) can be interpreted as the absence of *redundant assets*. Condition (R3) states that the discount factor ρ_t cannot be too close to 0 and 1. Under these assumptions, the following theorem holds.

Theorem 1. *The portfolio rule λ^* is a unique strategy forming a symmetric Nash equilibrium a.s. This equilibrium is strict.*

This result implies the following property of the portfolio rule λ^* . If all the investors except one, say investor i , use the strategy λ^* and i uses any other strategy λ distinct from λ^* , then the relative wealth $w_t^i / \sum_{j \neq i} w_t^j$ of i tends to zero at the exponential rate $\xi^i < 0$ (a.s.). In other words, the group of the Kelly investors drives the non-Kelly one out of the market, which is interpreted in evolutionary finance as the property of evolutionary stability of λ^* .

The result on the evolutionary stability of λ^* (without an exponential estimate of the convergence rate) follows from Theorem 1 in Evstigneev et al. (2008) requiring that the state space S is finite and all the strategies under consideration are completely mixed. It is important to note that results of this kind cannot, in general, be extended to settings going beyond the framework of the i.i.d. random parameters s_t . The reason for this lies in the fact that in more general settings, the analogues of the Kelly portfolio rule λ^* do not belong to the class of constant proportions strategies. They belong to broader classes of portfolio rules where there might be strategies “coexisting” with the generalized version of λ^* (strategies that cannot be driven by λ^* out of the market) — see Amir et al. (2009).

4 Proofs

A key role in the proof of Theorem 1 is played by a lemma which we formulate below. Consider a measurable vector function $R(s) = (R_1(s), \dots, R_K(s))$ on S satisfying (R1) and (R2). For any $\lambda = (\lambda_1, \dots, \lambda_K) \in \Delta^K$, $\kappa \in (0, 1]$ and $\rho' \leq \rho \leq \rho''$ (see (R3)), define

$$F_\rho(\lambda, \kappa; s) := \frac{\sum_{k=1}^K [\rho \lambda_k + (1 - \rho) R_k(s)] \frac{\lambda_k^*}{\lambda_k^* \kappa + \lambda_k (1 - \kappa)}}{\sum_{k=1}^K [\rho \lambda_k^* + (1 - \rho) R_k(s)] \frac{\lambda_k}{\lambda_k^* \kappa + \lambda_k (1 - \kappa)}}, \quad (14)$$

where $\lambda_k^* = ER_k(s)$. The function $F_\rho(\lambda, \kappa; s)$ is well-defined and takes on finite non-negative values.

Lemma 1. *For any $\lambda \in \Delta^K$ distinct from λ^* there exist constants $H > 0$ and $\delta > 0$ such that*

$$E \min\{H, \ln F_\rho(\lambda, \kappa; s)\} \geq \delta \quad (15)$$

for all $\kappa \in (0, 1]$ and all $\rho \in [\rho', \rho'']$.

The proof of this lemma is routine, but rather lengthy, and we relegate it to the Appendix.

For the proof of Theorem 1, we will need a description of a system of equations governing the dynamics of the *market shares*

$$r_t^i := w_t^i / \sum_j w_t^j$$

of the investors given their admissible strategy profile $(\lambda^1, \dots, \lambda^N)$. Consider the path (7) of the random dynamical system generated by $(\lambda^1, \dots, \lambda^N)$ and the sequence of vectors $r_t = (r_t^1, \dots, r_t^N)$ of the market shares of the investors at date t . It is proved in Amir et al. (2009), Proposition 2, that the following equations hold:

$$r_{t+1}^i = \sum_{k=1}^K [\rho_{t+1} \langle \lambda_k, r_{t+1} \rangle + (1 - \rho_{t+1}) R_{t+1,k}] \frac{\lambda_k^i r_t^i}{\langle \lambda_k, r_t \rangle}, \quad (16)$$

$i = 1, \dots, N, t \geq 0$.

In fact, we will show that in the proof of Theorem 1 it is sufficient to consider the case of two investors ($N = 2$). In this case, the analysis can be reduced to the consideration of the ratio

$$z_t := r_t^1 / r_t^2 = w_t^1 / w_t^2$$

of the market shares of investors 1 and 2 using the strategies $\lambda^* = (\lambda_1^*, \dots, \lambda_K^*)$ and $\lambda = (\lambda_1, \dots, \lambda_K)$. The dynamics of z_t are described by the following equation

$$z_{t+1} = z_t \frac{\sum_{k=1}^K [\rho_{t+1} \lambda_k + (1 - \rho_{t+1}) R_{t+1,k}] \frac{\lambda_k^*}{\lambda_k^* z_t + \lambda_k}}{\sum_{k=1}^K [\rho_{t+1} \lambda_k^* + (1 - \rho_{t+1}) R_{t+1,k}] \frac{\lambda_k}{\lambda_k^* z_t + \lambda_k}}. \quad (17)$$

For a proof of (17), see Amir et al. (2009), Proposition 4.

Proof of Theorem 1. To prove that λ^* forms a strict symmetric Nash equilibrium a.s. it is sufficient to consider the case of two investors 1 and 2, using λ^* and λ , and show that

$$\liminf_{t \rightarrow \infty} t^{-1} \ln z_t > 0 \text{ (a.s.)}, \quad (18)$$

where z_t is the ratio of the market shares of 1 and 2. To demonstrate that the problem reduces to the case of two investors, let us first observe that by virtue of symmetry, it is sufficient to verify the property (9) for $i = N$. Suppose investors $i = 1, 2, \dots, N-1$ use λ^* and investor N uses $\lambda \neq \lambda^*$. Then the total market share $r_t^* := r_t^1 + \dots + r_t^{N-1}$ of $i = 1, 2, \dots, N-1$ satisfies

$$r_{t+1}^* = \sum_{k=1}^K \left\{ \rho_{t+1} [\lambda_k^* r_{t+1}^* + \lambda_k r_{t+1}^N] + (1 - \rho_{t+1}) R_{t+1,k} \right\} \frac{\lambda_k^* r_t^*}{\lambda_k^* r_t^* + \lambda_k r_t^N}. \quad (19)$$

This relation is obtained by summing up equations (16) over $i = 1, 2, \dots, N-1$. At the same time, by virtue of (16), we have

$$r_{t+1}^N = \sum_{k=1}^K \left\{ \rho_{t+1} [\lambda_k^* r_{t+1}^* + \lambda_k r_{t+1}^N] + (1 - \rho_{t+1}) R_{t+1,k} \right\} \frac{\lambda_k^N r_t^N}{\lambda_k^* r_t^* + \lambda_k r_t^N}. \quad (20)$$

Thus the vector (r_t^*, r_t^N) evolves in time as the vector $(\tilde{r}_t^1, \tilde{r}_t^2)$ of market shares of two investors using the strategies λ^* and λ , respectively. If we know that (18) holds, then

$$\xi^N(\lambda^*, \dots, \lambda^*, \lambda) = \limsup_{t \rightarrow \infty} t^{-1} \ln \frac{w_t^N}{w_t^*} = \limsup_{t \rightarrow \infty} t^{-1} \ln \frac{r_t^N}{r_t^*} =$$

$$\limsup_{t \rightarrow \infty} (-t^{-1} \ln z_t) = -\liminf_{t \rightarrow \infty} (t^{-1} \ln z_t) < 0 = \xi^N(\lambda^*, \dots, \lambda^*, \lambda^*) \text{ (a.s.)},$$

where the last equality holds because the market shares of all the investors remain constant, as long as all of them use the same strategy (see (16)). This proves that λ^* forms a strict symmetric Nash equilibrium.

Let us verify (18). Put $G_t = \ln(z_t/z_{t-1})$. Then

$$\sum_{t=1}^T G_t = \sum_{t=1}^T (\ln z_t - \ln z_{t-1}) = \ln z_T - \ln z_0.$$

Therefore it suffices to prove that $\liminf_{T \rightarrow \infty} T^{-1} \sum_{t=1}^T G_t > 0$ (a.s.). For any constant H define $G_t^H := \min\{G_t, H\}$. Since $G_t^H \leq G_t$ it is sufficient to prove that

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T G_t^H > 0 \quad (\text{a.s.}) \quad (21)$$

for some H .

Observe that

$$\begin{aligned} G_{t+1} &= \ln \frac{z_{t+1}}{z_t} = \ln \frac{\sum_{k=1}^K [\rho_{t+1} \lambda_k + (1 - \rho_{t+1}) R_{t+1,k}] \frac{\lambda_k^*}{\lambda_k^* z_t + \lambda_k}}{\sum_{k=1}^K [\rho_{t+1} \lambda_k^* + (1 - \rho_{t+1}) R_{t+1,k}] \frac{\lambda_k}{\lambda_k^* z_t + \lambda_k}} = \\ &= \ln \frac{\sum_{k=1}^K [\rho_{t+1} \lambda_k + (1 - \rho_{t+1}) R_k(s_{t+1})] \frac{\lambda_k^*}{\lambda_k^* r_t^1 + \lambda_k(1 - r_t^1)}}{\sum_{k=1}^K [\rho_{t+1} \lambda_k^* + (1 - \rho_{t+1}) R_k(s_{t+1})] \frac{\lambda_k}{\lambda_k^* r_t^1 + \lambda_k(1 - r_t^1)}} = \ln F_{\rho_{t+1}}(\lambda, r_t^1; s_{t+1}), \end{aligned}$$

where $r_t^1 = r_t^1(s^t)$ and $\rho_{t+1} = \rho_{t+1}(s^t)$ (recall that the process ρ_t is predictable). By virtue of Lemma 1, there exist $H > 0$ and $\delta > 0$ such that $E_t G_{t+1}^H \geq \delta$, where $E_t(\cdot) = E(\cdot | s^t)$ is the conditional expectation given s^t and

$$G_{t+1}^H(s^{t+1}) = \min\{H, \ln F_{\rho_{t+1}(s^t)}(\lambda, r_t^1(s^t); s_{t+1})\}.$$

When computing $E_t G_{t+1}^H$ we fix s^t and take the unconditional expectation of G_{t+1}^H with respect to s_{t+1} , which is justified because s^t and s_{t+1} are independent.

Finally, we have

$$\frac{1}{T} \sum_{t=1}^T G_t^H = \frac{1}{T} \sum_{t=1}^T E_{t-1} G_t^H + \frac{1}{T} \sum_{t=1}^T (G_t^H - E_{t-1} G_t^H).$$

Since G_t^H is uniformly bounded, we can apply to the process $B_t^H := G_t^H - E_{t-1} G_t^H$ the strong law of large numbers for martingale differences (see, e.g., Hall and Heyde (1980)), which yields $\frac{1}{T} \sum_{t=1}^T B_t^H \rightarrow 0$ (a.s.). Therefore $\liminf T^{-1} \sum_{t=1}^T G_t^H \geq \delta$, which proves (21).

Suppose a strategy $\lambda \neq \lambda^*$ forms a symmetric Nash equilibrium with probability one. Then

$$0 = \xi^N(s^\infty; \lambda, \dots, \lambda) \geq \xi^N(s^\infty; \lambda, \dots, \lambda, \lambda^*) \quad (\text{a.s.}), \quad (22)$$

where

$$\xi^N(s^\infty; \lambda, \dots, \lambda, \lambda^*) = \limsup_{t \rightarrow \infty} t^{-1} \ln \frac{r_t^N}{1 - r_t^N}.$$

By interchanging λ and λ^* in formulas (19) and (20), we obtain that the vector $(r_t^1 + \dots + r_t^{N-1}, r_t^N)$ evolves in time as the vector $(\hat{r}_t^1, \hat{r}_t^2)$ of market shares of two investors using the strategies λ and λ^* , respectively. As we have proved above, this implies

$$\liminf_{t \rightarrow \infty} t^{-1} \ln \frac{r_t^N}{1 - r_t^N} > 0 \text{ (a.s.)}.$$

Thus $\xi^N(s^\infty; \lambda, \dots, \lambda, \lambda^*) > 0$, which yields the inequality " $<$ " in (22). This is a contradiction. \square

5 Appendix

Proof of Lemma 1. We first observe that the function $F_\rho(\lambda, \kappa; s)$ satisfies

$$F_\rho(\lambda, \kappa; s) \geq c^2, \quad (23)$$

where $c := \min_k \lambda_k^* (> 0)$. Indeed, $F_\rho(\lambda, \kappa; s) = A/B$, where $A \geq c$, and

$$\rho'c < B \leq c^{-1}. \quad (24)$$

Case 1. Assume that at least one of the coordinates of λ is zero, so that $\mathbf{K} := \{k : \lambda_k = 0\} \neq \emptyset$. Then

$$A = (1 - \rho) \frac{1}{\kappa} \sum_{k \in \mathbf{K}} R_k(s) + \sum_{k \notin \mathbf{K}} [\rho \lambda_k + (1 - \rho) R_k(s)] \frac{\lambda_k^*}{\lambda_k^* \kappa + \lambda_k (1 - \kappa)}$$

and

$$B = \sum_{k \notin \mathbf{K}} [\rho \lambda_k^* + (1 - \rho) R_k(s)] \frac{\lambda_k}{\lambda_k^* \kappa + \lambda_k (1 - \kappa)}.$$

By virtue of (R1), there exists $\theta > 0$ such that $\sum_{k \in \mathbf{K}} R_k(s) \geq \theta$ for all s in a set \bar{S} with $P(\bar{S}) > 0$. Therefore $d_1 \kappa^{-1} \mathbf{1}_{\bar{S}}(s) \leq A \leq \kappa^{-1} D_1$, where $\mathbf{1}_{\bar{S}}(s)$ is the indicator function of the set \bar{S} , $d_1 := (1 - \rho'')\theta$ and $D_1 := 1 + (\min_{k \notin \mathbf{K}} \lambda_k)^{-1}$. Also, we have $d_2 \leq B \leq D_2$, where $d_2 := \rho'c \min_{k \notin \mathbf{K}} \lambda_k$ and $D_2 := c^{-1}$ (see (24)). Thus

$$\kappa^{-1} d \mathbf{1}_{\bar{S}}(s) \leq F_\rho(\lambda, \kappa; s) \leq \kappa^{-1} D, \quad (25)$$

where $d := d_1/D_2$ and $D := D_1/d_2$. From the first of these inequalities and (23) we obtain $(\ln d - \ln \kappa) \mathbf{1}_{\bar{S}}(s) + (2 \ln c)(1 - \mathbf{1}_{\bar{S}}(s)) \leq \ln F_\rho(\lambda, \kappa; s)$, and so $E \min[H, \ln F_\rho(\lambda, \kappa; s)] \geq 2 \ln c + \min(H, \ln d - \ln \kappa) P(\bar{S})$.

Define

$$\bar{\kappa} := \exp\left[\ln d - \frac{1 - 2 \ln c}{P(\bar{S})}\right], \quad H := \ln D - \ln \bar{\kappa} \quad (26)$$

and observe that if $0 < \kappa < \bar{\kappa}$, then

$$E \min[H, \ln F_\rho(\lambda, \kappa; s)] \geq 2 \ln c + (\ln d - \ln \bar{\kappa})P(\bar{S}) = 1 \quad (27)$$

by virtue of the inequality $d \leq D$. If $\kappa \geq \bar{\kappa}$, then $\ln F_\rho(\lambda, \kappa; s) \leq \ln D - \ln \kappa \leq \ln D - \ln \bar{\kappa} = H$, and so $\min[H, \ln F_\rho(\lambda, \kappa; s)] = \ln F_\rho(\lambda, \kappa; s)$. Thus in order to complete the proof of the lemma in the case when λ has zero coordinates it remains to show that

$$\inf_{\kappa \in [\bar{\kappa}, 1], \rho \in [\rho', \rho'']} E \ln F_\rho(\lambda, \kappa; s) > 0 \quad (28)$$

for each $\bar{\kappa} \in (0, 1]$. Indeed, then H can be defined by (26) and δ can be defined as the minimum of 1 (see (27)) and the infimum in (28).

By virtue of (23) and (25), the function $E \ln F_\rho(\lambda, \kappa; s)$ is continuous on the compact set $[\rho', \rho''] \times [\bar{\kappa}, 1]$ and hence it attains its minimum on this set. Thus, in order to establish (28) it is sufficient to prove that $E \ln F_\rho(\lambda, \kappa; s) > 0$ for each $\rho \in [0, 1]$ and $\kappa \in (0, 1]$.

By applying Jensen's inequality, we find

$$\begin{aligned} & E \ln \sum_{k=1}^K [\rho \lambda_k + (1 - \rho) R_k(s)] \frac{\lambda_k^*}{\lambda_k^* \kappa + \lambda_k (1 - \kappa)} \geq \\ & \rho \ln \sum_{k=1}^K \frac{\lambda_k \lambda_k^*}{\lambda_k^* \kappa + \lambda_k (1 - \kappa)} + (1 - \rho) E \ln \sum_{k=1}^K R_k(s) \frac{\lambda_k^*}{\lambda_k^* \kappa + (1 - \kappa) \lambda_k} \geq \\ & \rho \ln \sum_{k=1}^K \frac{\lambda_k \lambda_k^*}{\lambda_k^* \kappa + \lambda_k (1 - \kappa)} + (1 - \rho) \sum_{k=1}^K \lambda_k^* \ln \frac{\lambda_k^*}{\lambda_k^* \kappa + (1 - \kappa) \lambda_k}, \end{aligned} \quad (29)$$

and

$$\begin{aligned} & E \ln \sum_{k=1}^K [\rho \lambda_k^* + (1 - \rho) R_k(s)] \frac{\lambda_k}{\lambda_k^* \kappa + \lambda_k (1 - \kappa)} < \\ & \ln \left\{ \sum_{k=1}^K E [\rho \lambda_k^* + (1 - \rho) R_k(s)] \frac{\lambda_k}{\lambda_k^* \kappa + \lambda_k (1 - \kappa)} \right\} = \ln \sum_{k=1}^K \frac{\lambda_k \lambda_k^*}{\lambda_k^* \kappa + \lambda_k (1 - \kappa)}. \end{aligned} \quad (30)$$

The inequality in (30) is strict because there is no constant γ such that

$$\sum_{k=1}^K [\rho \lambda_k^* + (1 - \rho) R_k(s)] \frac{\lambda_k}{\lambda_k^* \kappa + \lambda_k (1 - \kappa)} = \gamma \text{ (a.s.)}. \quad (31)$$

Indeed, if (31) holds, then

$$\sum_{k=1}^K [\rho \lambda_k^* + (1 - \rho) R_k(s)] \gamma_k = 0 \text{ (a.s.)}, \quad (32)$$

where $\gamma_k := \lambda_k [\lambda_k^* \kappa + \lambda_k (1 - \kappa)]^{-1} - \gamma$. Observe that at least one of the numbers γ_k is not equal to zero. Otherwise $\lambda_k = \gamma [\lambda_k^* \kappa + \lambda_k (1 - \kappa)]$ for all k , and by summing these equalities over k , we get $\gamma = 1$, which yields $\lambda_k = \lambda_k^* \kappa + \lambda_k (1 - \kappa)$, $\lambda_k^* \kappa = \lambda_k \kappa$, and $\lambda_k^* = \lambda_k$ (recall that $\kappa \neq 0$). This is a contradiction because $\lambda \neq \lambda^*$. Thus $\gamma = (\gamma_1, \dots, \gamma_K) \neq 0$, and $\sum_{k=1}^K R_k(s) \gamma_k = b$ (a.s.), where b is some constant. This constant is not zero because the functions $R_k(s)$ are linearly independent. By setting $\gamma'_k := \gamma_k / b$, we obtain that the non-zero vector $\gamma' = (\gamma'_1, \dots, \gamma'_K)$ satisfies $\sum_{k=1}^K R_k(s) \gamma'_k = 1$ (a.s.), which yields $\sum_{k=1}^K R_k(s) (\gamma'_k - 1) = 0$ (a.s.). In view of the linear independence of $R_k(s)$, this implies $\gamma'_1 = \dots = \gamma'_K = 1$. Since $\gamma_k = b \gamma'_k = b$, we obtain that the left-hand side of (32) is equal to $b \neq 0$, which is a contradiction.

From (29) and (30) we get

$$E \ln F_\rho(\lambda, \kappa; s) > (1 - \rho) \left[\sum_{k=1}^K \lambda_k^* \ln \frac{\lambda_k^*}{\lambda_k^* \kappa + (1 - \kappa) \lambda_k} - \ln \sum_{k=1}^K \frac{\lambda_k \lambda_k^*}{\lambda_k^* \kappa + \lambda_k (1 - \kappa)} \right]. \quad (33)$$

Denote the expression in the square brackets in (33) by $\Phi_\kappa(\lambda)$. It is proved in Evstigneev et al., pp. 337-338, that

$$\Phi_\kappa(\lambda) \geq 0 \text{ for each } \kappa \in [0, 1] \quad (34)$$

if $\lambda > 0$. Therefore $\Phi_\kappa(\lambda(1 - \varepsilon) + \varepsilon \lambda^*) > 0$ for each $\varepsilon > 0$. The function $\Phi_\kappa(\lambda)$ is finite and continuous on Δ^K (because $\lambda_k^* > 0$ and $\kappa > 0$). Consequently, $\Phi_\kappa(\lambda) = \lim_{\varepsilon \downarrow 0} \Phi_\kappa(\lambda(1 - \varepsilon) + \varepsilon \lambda^*) \geq 0$. By using (33), we obtain that $E \ln F_\rho(\lambda, \kappa) > 0$ for all $\rho \in [0, 1)$ and $\kappa \in (0, 1]$. This completes the proof of the lemma in the case when the vector λ has zero coordinates.

Case 2. Now assume that $\lambda_k > 0$ for each k . Then the function $\ln F_\rho(\lambda, \kappa; s)$ is well-defined, finite, continuous with respect to (ρ, κ) on the set $[\rho', \rho''] \times [0, 1]$ (including $\kappa = 0$) and uniformly bounded:

$$2 \ln c \leq \ln F_\rho(\lambda, \kappa; s) \leq \ln(\min_k \lambda_k)^{-2}.$$

To complete the proof it is sufficient to show that the infimum in (28) with $\bar{\kappa} = 0$ is strictly positive (then δ can be defined as this infimum and H as $2|\ln c| + 2|\ln \min \lambda_k|$). In view of the continuity of $E \ln F_\rho(\lambda, \kappa; s)$ this will be proved if we establish the inequality $E \ln F_\rho(\lambda, \kappa; s) > 0$ for each $\rho \in [0, 1)$

and $\kappa \in [0, 1]$. If $\kappa > 0$, this inequality is proved by exactly the same arguments as above – by deriving relations (29), (30), (33) and using (34). If $\kappa = 0$, we change the above arguments as follows: instead of strict, we establish non-strict inequalities in (30) and show that the right-hand side of (33) is strictly positive, because $\Phi_0(\lambda) = \sum_{k=1}^K \lambda_k^* \ln(\lambda_k^*/\lambda_k) > 0$.

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