# Water waves in the time domain

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14 April 2010

**Abstract**

In this paper, the initial value problem for a structure floating on the surface of the sea is investigated under the assumptions of linear theory. Fourier transforms are used to connect the time- and frequency-domain representations of the coupled motion of the fluid and body. This allows the large-time asymptotics of the motion to be obtained from the singularity structure of the frequency-domain potential in the complex plane. Under certain initial conditions, the free motion of a body about a fixed, equilibrium position is shown not to exist for all time, and in this case the assumptions behind the linear theory are violated. For suitably moored structures, motion is found which is purely exponentially decaying in time and does not involve any oscillations.

*Dedicated to the memory of Ernie Tuck in appreciation of his life and work.*

## 1 Introduction

The description of the motion of a floating structure using the linearised equations of water waves is now very well established. In general, published papers in this area concern either the time-domain response to particular initial conditions or, most commonly, time-harmonic motions of the fluid and structure which are assumed to have persisted for all time. The two are linked, at least in part, through a Fourier transform in time. Fourier transformation of the time-domain problem yields an equivalent frequency-domain problem which, if it can be solved for all frequencies, yields the solution to the original problem through the inverse Fourier transform. In general, the frequency-domain problem obtained by Fourier transformation involves the initial conditions from the time-domain problem so that care must be taken in the interpretation of purely time-harmonic motions (in this paper "time-harmonic motions" will refer specifically to motions in the frequency domain for which the initial conditions do not appear in the governing equations). Although there is an extensive literature on this subject published over many years (reviewed, for example, in [1, 2, 3]), to the authors' knowledge a detailed account of the theory that incorporates the initial conditions in to the frequency-domain problem is not available and part of the purpose of the present paper is to provide such an account (see sections 2 and 3). In a related approach developed by Cummins [4], the time-dependent motion of a floating structure may be studied using a system of integro-differential equations that involve quantities, such as the added mass and damping coefficients, that are obtained from study of the time-harmonic problem. These equations were derived originally through consideration of the responses to impulsive motions of the structure; in section 4 of the present paper it is shown how the equations follow, in a straightforward way, from the inverse Fourier transform of the full frequency-domain equations of motion for the structure that incorporate the initial conditions. A difficulty with the Fourier-transform approach is that the solution to a frequency-domain problem corresponding to a specified initial disturbance of the fluid is not straightforward. However, for a fixed structure in the presence of an initially-disturbed fluid, recent work on the expansion of water-wave potentials in terms of generalised eigenfunctions shows how the time-domain solution may be expressed in terms of the solutions to time-harmonic scattering problems [5, 6].

One advantage of the Fourier transform approach to the initial-value problem is that large-time asymptotics are obtained readily from knowledge of the corresponding frequency-domain problem. The time-domain solution is expressed as an inverse Fourier transform in which the path of integration goes along the real axis in the complex frequency domain, and over any singularities of the frequency-domain solution that lie on the real axis. If the path of integration is then moved downwards in to the lower half plane, contributions to the large-time asymptotics are obtained from any singularities in the frequency-domain solution that are encountered during this process; the method is explained here in section 5. The singularities that most commonly arise are: simple poles on the real axis that correspond to persistent oscillations; a branch point at the origin that corresponds to algebraic decay of the motion [7, 8]; simple poles in the lower half plane (sometimes called "complex resonances") that correspond to exponentially decaying oscillations [9, 10, 11]. In addition, the translational and rotational behaviour arising from simple and double poles at the origin is discussed here.

In section 6 of the paper, it is shown that solutions of the time-domain problem are possible with purely exponential decay of the motion of both the fluid and the structure, but for which there are *no* oscillations. A solution of this type is constructed for the vertical motion of a restrained surface-piercing structure in water of infinite depth. For most initial conditions, and in the absence of persistent oscillatory forcing, the time-dependent motion of such a structure would be dominated by a decaying oscillation followed by an ultimate algebraic decay to rest. However, by correct choice of the initial conditions it is possible to obtain a frequency-domain solution for which the only singularity is a simple pole on the negative imaginary axis, and consequently the time dependence in the time-domain solution is a separable exponential decay. This type of solution appears to be new in the water-wave problem.

In section 7, the circumstances under which persistent oscillations may be obtained are discussed, and some consequences of the asymptotic results obtained in section 5 are explored. An investigation is made of the ultimate algebraic decay that may arise when a structure, able to move vertically, is released from rest and, in particular, a new result for a three-dimensional structure in water of finite depth is obtained. It is then shown that an unrestrained asymmetric structure that is displaced vertically and released from rest will, in general, undergo a horizontal translation proportional to the initial vertical displacement. Finally, the effect of forces on an unrestrained structure is discussed and it is shown that, in certain circumstances, a structure will be given a horizontal velocity that leads to a violation of the assumptions behind the linearised theory. The methods described here should allow other situations to be investigated with ease.

## 2 The initial-value problem

An inviscid and incompressible fluid with a free surface is contained within a horizontal layer of depth $h$ that is bounded below by a rigid bed and extends to infinity in all horizontal directions. Cartesian coordinates $(x, y, z)$ are chosen with $z$ measured vertically upwards and the origin in the mean free surface. The fluid layer contains a floating structure that is free to move in all six rigid-body modes of motion and a displacement in mode $p$ is denoted by $X_p(t)$, $p = 1..6$. Modes 1, 2 and 3 respectively correspond to translational motions in the $x$, $y$ and $z$ directions and are known as surge, sway and heave respectively, while 4, 5 and 6 are the corresponding rotational modes, respectively known as roll, pitch and yaw (and therefore each of $X_p$, $p = 4..6$, is an angle). The wetted surface of the structure is denoted by $S_B$, a normal coordinate to $S_B$ directed out of the fluid is denoted by $n$, and $n_p$ is the $p$ component of the generalized unit normal to $S_B$ (see [12, section 1.3.3] for the definitions of the normal components for rotational modes).

The displacements of the fluid and structure from their initial states are assumed to be small relative to the other length scales in the problem. In addition, the fluid motion is assumed to be irrotational so that it may be described by a velocity potential $\Phi(\mathbf{x}, z, t)$, $\mathbf{x} = (x, y)$, that satisfies

$$\nabla^2 \Phi = 0 \tag{1}$$

within the fluid region $\mathcal{D}$ and the bed condition

$$\frac{\partial \Phi}{\partial z} = 0 \quad \text{on} \quad z = -h. \tag{2}$$

If $h$ is sufficiently large compared to the structure dimensions so that the fluid may be assumed to have infinite depth, (2) is replaced by

$$|\nabla \Phi| \to 0 \quad \text{as} \quad z \to -\infty. \tag{3}$$

On the mean free surface $S_F$ (that part of $z = 0$ exterior to any structures) the velocity potential is related to the surface elevation $H(\mathbf{x}, t)$ through the conditions

$$\frac{\partial \Phi}{\partial z} = \frac{\partial H}{\partial t} \quad \text{on} \quad S_F \tag{4}$$

and

$$H = -\frac{1}{g}\frac{\partial \Phi}{\partial t} \quad \text{on} \quad S_F, \tag{5}$$

which may be combined to yield

$$\frac{\partial^2\Phi}{\partial t^2} + g\frac{\partial\Phi}{\partial z} = 0 \quad \text{on} \quad S_F, \tag{6}$$

where $g$ is the acceleration due to gravity. On the structure the normal fluid velocity must match that of the structure so that

$$\frac{\partial\Phi}{\partial n} = \sum_{p=1}^{6} V_p(t)\, n_p \quad \text{on} \quad S_B \tag{7}$$

where $V_p(t) \equiv \dot{X}_p(t)$. In addition it is required that at any finite time

$$|\nabla\Phi| \to 0 \quad \text{as} \quad |\mathbf{x}| \to \infty \tag{8}$$

and the motion is subject to the initial conditions

$$\Phi(\mathbf{x}, 0, 0) = \Phi_0(\mathbf{x}), \quad H(\mathbf{x}, 0) = H_0(\mathbf{x}), \quad \mathbf{x} \in S_F, \tag{9}$$

where $\Phi_0(\mathbf{x})$ and $H_0(\mathbf{x})$ are prescribed; the second initial condition is equivalent to

$$\frac{\partial\Phi}{\partial t}(\mathbf{x}, 0, 0) = -g\, H_0(\mathbf{x}), \quad \mathbf{x} \in S_F. \tag{10}$$

The structure is taken to be moored by an arrangement of linear springs and dampers so that the equations of motion for the structure are

$$\sum_{q=1}^{6} M_{pq}\dot{V}_q(t) = -\rho \iint_{S_B} \frac{\partial\Phi}{\partial t}(\mathbf{x}, z, t)\, n_p \,\mathrm{d}S - \sum_{q=1}^{6} [c_{pq}X_q(t) + \gamma_{pq}V_q(t)] + F_p(t), \quad p = 1..6; \tag{11}$$

see [3, section 8.2.4]. Here $M_{pq}$ are the elements of the mass matrix, $\rho$ is the density of the fluid, and the constants $c_{pq}$ and $\gamma_{pq}$ describe the characteristics of respectively the springs and dampers in the moorings; appropriate buoyancy contributions are included in the spring coefficients (only heave, roll and pitch motions induce non-zero buoyancy forces). The first term on the right-hand side of (11) is the hydrodynamic force arising from the fluid motion, and $F_p(t)$ is an external force (not related to the fluid motion or the moorings) applied in mode $p$. In addition, for each $p$ the initial displacement $X_p(0)$ and velocity $V_p(0)$ of the structure must be prescribed.

# 3 The frequency-domain problem

## 3.1 The complete problem

The frequency-domain problem is obtained by Fourier transformation of the time-domain problem described in section 2. For a function $F(t)$ that is piecewise smooth and satisfies

$$F(t) = 0, \quad t < 0, \tag{12}$$

its corresponding Fourier transform is

$$f(\omega) = \int_0^\infty F(t)\,\mathrm{e}^{\mathrm{i}\omega t}\ \mathrm{d}t \equiv \mathcal{F}\{F(t)\}, \quad \operatorname{Im}\omega = v > 0, \tag{13}$$

and the inversion formula is

$$F(t) = \frac{1}{2\pi}\int_{-\infty+\mathrm{i}v}^{\infty+\mathrm{i}v} f(\omega)\,\mathrm{e}^{-\mathrm{i}\omega t}\ \mathrm{d}\omega \equiv \mathcal{F}^{-1}\{f(\omega)\}. \tag{14}$$

In general, the frequency parameter $\omega$ has a positive imaginary part [13, section 5.6] to allow for the possibility that $F(t)$ may be oscillatory, or grow algebraically, as time $t \to \infty$. The integration path in equation (14) can be moved onto the real $\omega$ axis as long as it passes over any singularities that may lie on the axis. Furthermore, in the frequency domain the transformed equations may be continued analytically onto the real $\omega$ axis with the possible exception of any isolated singular points. From the definition of the Fourier transform, the dimension of each frequency-domain quantity has an additional factor of time multiplying the dimension of the corresponding time-domain quantity.

The frequency-domain potential $\phi(\mathbf{x}, z, \omega)$ is the Fourier transform of the time-domain potential $\Phi(\mathbf{x}, z, t)$ and satisfies

$$\nabla^2\phi = 0 \quad \text{in} \quad \mathcal{D} \tag{15}$$

and the bed condition

$$\frac{\partial\phi}{\partial z} = 0 \quad \text{on} \quad z = -h. \tag{16}$$

On the free surface the velocity potential is related to the transform $\eta(\mathbf{x}, \omega)$ of the surface elevation $H(\mathbf{x}, t)$ through the transformed free-surface conditions

$$\frac{\partial\phi}{\partial z} = -\mathrm{i}\omega\eta - H_0 \quad \text{on} \quad S_F \tag{17}$$

and

$$\eta = \frac{1}{g}\left(\mathrm{i}\omega\phi + \Phi_0\right) \quad \text{on} \quad S_F, \tag{18}$$

which may be combined to yield

$$\frac{\partial\phi}{\partial z} = \frac{\omega^2}{g}\phi - \frac{\mathrm{i}\omega}{g}\Phi_0 - H_0 \quad \text{on} \quad S_F. \tag{19}$$

The transformed boundary condition on the structure is

$$\frac{\partial\phi}{\partial n} = \sum_{p=1}^{6} v_p\, n_p \quad \text{on} \quad S_B \tag{20}$$

where

$$v_p(\omega) = -\mathrm{i}\omega x_p(\omega) - X_p(0) \tag{21}$$

is the transform of the velocity of the structure, $V_p(t)$. In addition, the frequency-domain velocity potential must satisfy an appropriately formulated radiation condition. Fourier transformation of the time-domain equations of motion (11) yields the frequency-domain equations of motion

$$\sum_{q=1}^{6}\left[c_{pq} - \mathrm{i}\omega\gamma_{pq} - \omega^2 M_{pq}\right] v_q(\omega) = \omega^2\rho \iint_{S_B} \phi(\mathbf{x}, z, \omega)\, n_p\, \mathrm{d}S$$

$$- \mathrm{i}\omega\rho \iint_{S_B} \Phi(\mathbf{x}, z, 0^+)\, n_p\, \mathrm{d}S - \sum_{q=1}^{6}\left[c_{pq}X_q(0) + \mathrm{i}\omega M_{pq}V_q(0)\right] - \mathrm{i}\omega f_p(\omega), \quad p = 1..6. \quad (22)$$

Even when the fluid around the structure is initially at rest, for a motion of the structure that begins with a non-zero velocity then

$$\Phi(\mathbf{x}, z, 0^+) \equiv \lim_{t\to 0^+} \Phi(\mathbf{x}, z, t) \neq 0, \quad (\mathbf{x}, z) \in \mathcal{D}; \quad (23)$$

see equation (34) below.

Because of the appearance of the initial conditions in the free-surface condition (19), the above problem is equivalent to the problem of wave generation by an oscillatory pressure applied to the free surface. In the absence of a structure, problems of this type are discussed by Wehausen & Laitone [1, section 21].

## 3.2 The scattering and radiation problems

The frequency-domain potential is decomposed by writing

$$\phi(\mathbf{x}, z, \omega) = \phi_{\mathrm{S}}(\mathbf{x}, z, \omega) + \phi_{\mathrm{R}}(\mathbf{x}, z, \omega). \quad (24)$$

Here $\phi_{\mathrm{S}}(\mathbf{x}, z, \omega)$ is the flow field in the presence of the fixed structure that results from the initial conditions in the free-surface condition and is the solution of the problem $\mathcal{P}_{\mathrm{S}}$ given by

$$\left.\begin{aligned}
&\nabla^2\phi_{\mathrm{S}} = 0 \quad \text{in} \quad \mathcal{D},\\
&\frac{\partial\phi_{\mathrm{S}}}{\partial z} = 0 \quad \text{on} \quad z = -h,\\
&\frac{\partial\phi_{\mathrm{S}}}{\partial z} = \frac{\omega^2}{g}\phi_{\mathrm{S}} - \frac{\mathrm{i}\omega}{g}\Phi_0 - H_0 \quad \text{on} \quad S_F,\\
&\frac{\partial\phi_{\mathrm{S}}}{\partial n} = 0 \quad \text{on} \quad S_B,
\end{aligned}\right\} \quad (25)$$

together with an appropriately-formulated radiation condition. For non-zero initial conditions, problem $\mathcal{P}_{\mathrm{S}}$ is *not* that solved in the consideration of time-harmonic motions. The potential $\phi_{\mathrm{R}}(\mathbf{x}, z, \omega)$ is the radiation potential that describes the fluid motion when the structure is forced to oscillate in the absence of any waves incident upon the structure and it is the solution of the problem

$$\left.\begin{aligned}
&\nabla^2\phi_{\mathrm{R}} = 0 \quad \text{in} \quad \mathcal{D},\\
&\frac{\partial\phi_{\mathrm{R}}}{\partial z} = 0 \quad \text{on} \quad z=-h,\\
&\frac{\partial\phi_{\mathrm{R}}}{\partial z} = \frac{\omega^2}{g}\phi_{\mathrm{S}} \quad \text{on} \quad S_F,\\
&\frac{\partial\phi_{\mathrm{R}}}{\partial n} = \sum_{p=1}^{6} v_p(\omega)\, n_p \quad \text{on} \quad S_B,
\end{aligned}\right\} \tag{26}$$

together with a radiation condition that specifies outgoing waves. In a further decomposition, this total radiation potential is written

$$\phi_{\mathrm{R}}(\mathbf{x}, z, \omega) = \sum_{p=1}^{6} v_p(\omega)\phi_p(\mathbf{x}, z, \omega), \tag{27}$$

where each modal radiation potential $\phi_p$ satisfies a particular radiation problem $\mathcal{P}_p$ given by

$$\left.\begin{aligned}
&\nabla^2\phi_p = 0 \quad \text{in} \quad \mathcal{D},\\
&\frac{\partial\phi_p}{\partial z} = 0 \quad \text{on} \quad z=-h,\\
&\frac{\partial\phi_p}{\partial z} = \frac{\omega^2}{g}\phi_p \quad \text{on} \quad S_F,\\
&\frac{\partial\phi_p}{\partial n} = n_p \quad \text{on} \quad S_B,
\end{aligned}\right\} \tag{28}$$

together with a radiation condition specifying outgoing waves. The problem $\mathcal{P}_p$ is identical to that solved in studies of time-harmonic motions.

Once the solutions to $\mathcal{P}_{\mathrm{S}}$ and each $\mathcal{P}_p$ are known, and each velocity $v_p$ has been determined by solution of the equations of motion (22), the time domain potential $\Phi(\mathbf{x}, z, t)$ can be recovered by inverse Fourier transform. It should be noted that for $\omega \in \mathbb{R}$ the inverse Fourier transform of a term containing a modal radiation potential cannot be computed directly as $\phi_p$ does not tend to zero as $\omega \to \infty$. Specifically,

$$\phi_p(\mathbf{x}, z, \omega) \sim \Omega_p(\mathbf{x}, z) \quad \text{as} \quad \omega \to \infty \tag{29}$$

where the infinite-frequency radiation potential $\Omega_p(\mathbf{x}, z)$ is the solution of

$$\left.\begin{aligned}
&\nabla^2\Omega_p = 0 \quad \text{in} \quad \mathcal{D},\\
&\frac{\partial\Omega_p}{\partial z} = 0 \quad \text{on} \quad z=-h,\\
&\Omega_p = 0 \quad \text{on} \quad S_F,\\
&\frac{\partial\Omega_p}{\partial n} = n_p \quad \text{on} \quad S_B,\\
&\Omega_p \to 0 \quad \text{as} \quad |\mathbf{x}| \to \infty.
\end{aligned}\right\} \tag{30}$$

Thus, the time-domain potential

$$\Phi(\mathbf{x},z,t) = \mathcal{F}^{-1}\{\phi_{\mathrm{S}}(\mathbf{x},z,\omega)\} + \sum_{p=1}^{6}\Big[\mathcal{F}^{-1}\left\{v_p(\omega)\left[\phi_p(\mathbf{x},z,\omega) - \Omega_p(\mathbf{x},z)\right] + v_p(\omega)\Omega_p(\mathbf{x},z)\right\}\Big]$$

$$= \Phi_{\mathrm{S}}(\mathbf{x},z,t) + \sum_{p=1}^{6}\left[\int_{-\infty}^{\infty} V_p(\tau)\Gamma_p(\mathbf{x},z,t-\tau)\,\mathrm{d}\tau + V_p(t)\Omega(\mathbf{x},z)\right] \tag{31}$$

where the convolution theorem for Fourier transforms has been used and

$$\Gamma_p(\mathbf{x},z,t) = \mathcal{F}^{-1}\left\{\phi_p(\mathbf{x},z,\omega) - \Omega_p(\mathbf{x},z)\right\}. \tag{32}$$

As there is no motion for $t < 0$, the convolution integral may be simplified to get

$$\Phi(\mathbf{x},z,t) = \Phi_{\mathrm{S}}(\mathbf{x},z,t) + \sum_{p=1}^{6}\left[\int_{0}^{t} V_p(\tau)\Gamma_p(\mathbf{x},z,t-\tau)\,\mathrm{d}\tau + V_p(t)\Omega(\mathbf{x},z)\right] \tag{33}$$

and hence

$$\Phi(\mathbf{x},z,0^+) = \Phi_{\mathrm{S}}(\mathbf{x},z,0) + \sum_{p=1}^{6} V_p(0)\Omega(\mathbf{x},z). \tag{34}$$

The hydrodynamic force in the frequency-domain equation of motion (22) can be expressed in terms of the components of the exciting force

$$\xi_p(\omega) = \mathrm{i}\omega\rho\iint_{S_B}\phi_{\mathrm{S}}(\mathbf{x},z,\omega)\,n_p\,\mathrm{d}S \tag{35}$$

and the added mass coefficients $\mu_{pq}$ and damping coefficients $\nu_{pq}$ defined through

$$\varpi_{pq}(\omega) \equiv \mu_{pq}(\omega) + \mathrm{i}\nu_{pq}(\omega) = \rho\iint_{S_B}\phi_q(\mathbf{x},z,\omega)\,n_p\,\mathrm{d}S. \tag{36}$$

In view of (34)

$$\rho\iint_{S_B}\Phi(\mathbf{x},z,0^+)\,n_p\,\mathrm{d}S = \rho\iint_{S_B}\Phi_{\mathrm{S}}(\mathbf{x},z,0)\,n_p\,\mathrm{d}S + \rho\sum_{q=1}^{6}V_q(0)\iint_{S_B}\Omega_q(\mathbf{x},z)\,n_p\,\mathrm{d}S$$

$$= \widetilde{\Phi}_{\mathrm{S},p} + \sum_{q=1}^{6}V_q(0)\mu_{pq}(\infty). \tag{37}$$

where

$$\widetilde{\Phi}_{\mathrm{S},p} = \rho\iint_{S_B}\Phi_{\mathrm{S}}(\mathbf{x},z,0)\,n_p\,\mathrm{d}S. \tag{38}$$

By properties of Fourier integrals, and provided $\Phi_{\mathrm{S}}(\mathbf{x},z,t)$ is suitably smooth [14, section 3.2],

$$\widetilde{\Phi}_{\mathrm{S},p} = -\lim_{|\omega|\to\infty}\xi_p(\omega) \tag{39}$$

for constant $\operatorname{Im}\omega$.

With the above definitions, the frequency-domain equations of motion (22) may be rewritten as

$$\sum_{q=1}^{6}\Big[c_{pq} - \mathrm{i}\omega\gamma_{pq} - \omega^2(M_{pq} + \mu_{pq}(\omega) + \mathrm{i}\nu_{pq}(\omega))\Big]v_q(\omega)$$

$$= -\sum_{q=1}^{6}\Big[c_{pq}X_q(0) + \mathrm{i}\omega(M_{pq} + \mu_{pq}(\infty))V_q(0)\Big] - \mathrm{i}\omega\widetilde{\Phi}_{\mathrm{S},p} - \mathrm{i}\omega\left[\xi_p(\omega) + f_p(\omega)\right], \quad p = 1..6. \quad (40)$$

The added-mass and damping coefficients, respectively $\mu_{pq}(\omega)$ and $\nu_{pq}(\omega)$, depend on the solution of particular radiation problems for time-harmonic motion with frequency $\omega$. However, because of the appearance of the initial conditions in the free-surface condition for the frequency-domain scattering potential, see equations (25), the exciting force $\xi_p(\omega)$ is not equivalent to a time-harmonic force.

## 4 The equations of Cummins

By considering the response to an impulsive motion of the structure, Cummins [4] devised time-domain equations of motion for a structure based on knowledge of the added mass and damping coefficients and the exciting force (see also [3, section 8.12]). Here the equations of Cummins are obtained directly from the frequency-domain equations of motion (40). A rearrangement of equations (40) gives, with the aid of (21),

$$\sum_{q=1}^{6}\Big[(M_{pq} + \mu_{pq}(\infty))(-V_q(0) - \mathrm{i}\omega v_q(\omega)) + l_{pq}(\omega)(-V_q(0) - \mathrm{i}\omega v_q(\omega)) + V_q(0)l_{pq}(\omega)$$

$$+ \; c_{pq}x_q(\omega) + \gamma_{pq}v_q(\omega)\Big] = \widetilde{\Phi}_{\mathrm{S},p} + \xi_p(\omega) + f_p(\omega), \quad p = 1..6, \quad (41)$$

where

$$l_{pq}(\omega) = \mu_{pq}(\omega) - \mu_{pq}(\infty) + \mathrm{i}\nu_{pq}(\omega). \quad (42)$$

Application of the inverse Fourier transform and the convolution theorem gives the Cummins' equations

$$\sum_{q=1}^{6}\Big[(M_{pq} + \mu_{pq}(\infty))\dot{V}_q(t) + \int_0^t L_{pq}(t-\tau)\dot{V}_q(\tau)\,\mathrm{d}\tau + V_q(0)L_{pq}(t)$$

$$+ \; c_{pq}X_q(t) + \gamma_{pq}V_q(t)\Big] = \Xi_p(t) + F_p(t), \quad p = 1..6, \quad (43)$$

where the time-domain exciting force arising from the scattering of the incident wave is

$$\Xi_p(t) = -\rho\iint_{S_B}\frac{\partial\Phi_{\mathrm{S}}}{\partial t}(\mathbf{x}, z, t)\,n_p\,\mathrm{d}S. \quad (44)$$

and the so-called impulse response function

$$L_{pq}(t) = \mathcal{F}^{-1}\{l_{pq}(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \big[\mu_{pq}(\omega) - \mu_{pq}(\infty) + \mathrm{i}\nu_{pq}(\omega)\big]\, \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{d}\omega \tag{45}$$

(more conveniently for computation, $L_{pq}(t)$ may be written as the inverse Fourier cosine transform of $\mu_{pq}(\omega) - \mu_{pq}(\infty)$, or the inverse Fourier sine transform of $\nu_{pq}(\omega)$ [2]). Note that

$$\mathcal{F}\{\Xi_p(t)\} = -\rho \iint_{S_B} \Big[-\mathrm{i}\omega\phi_{\mathrm{S}}(\mathbf{x}, z, \omega) - \Phi_{\mathrm{S}}(\mathbf{x}, z, 0)\Big] n_p\, \mathrm{d}S = \xi_p(\omega) + \widetilde{\Phi}_{\mathrm{S},p}. \tag{46}$$

A reduction of equation (43) for unrestrained heave motion in the absence of incident waves is given by [15].

With terms added here to represent damping in the moorings, [3, equation 8.12.32] gives the Cummins' equations in the form

$$\sum_{q=1}^{6}\Big[(M_{pq} + \mu_{pq}(\infty))\ddot{X}_q(t) + \int_{-\infty}^{t} L_{pq}(t-\tau)\ddot{X}_q(\tau)\,\mathrm{d}\tau \\ + \; c_{pq}X_q(t) + \gamma_{pq}\dot{X}_q(t)\Big] = \Xi_p(t) + F_p(t), \quad p = 1..6, \tag{47}$$

for motion that has persisted for all times until 'the present'. In the case that there is no motion for $t < 0$ then

$$\dot{X}_p(t) = V_p(t)U(t), \tag{48}$$

so that

$$\ddot{X}_p(t) = \dot{V}_p(t)U(t) + V_p(t)\delta(t), \tag{49}$$

where $U(t)$ and $\delta(t)$ are respectively the unit step function and the delta function; equation (47) then reduces to equation (43).

## 5 Large-time asymptotics of a Fourier integral

Consider the Fourier integral

$$F(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\omega)\, \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{d}\omega \tag{50}$$

where the path of integration passes over any singularities of $f$ that lie on the real $\omega$ axis as in figure 1 (see equation (14) and the remarks following it). It is straightforward to make deductions about the large-time asymptotics of $F(t)$ from the singularity structure of its Fourier transform $f(\omega)$ in the lower half of the complex $\omega$ plane, including the real axis. Here consideration will be given to poles and branch points; for the latter the required cuts are taken downwards in the

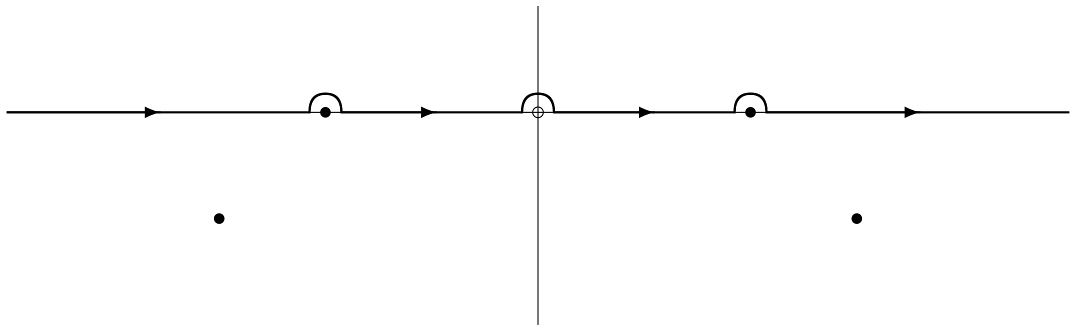

Figure 1: The integration path for the Fourier integral in equation (50).

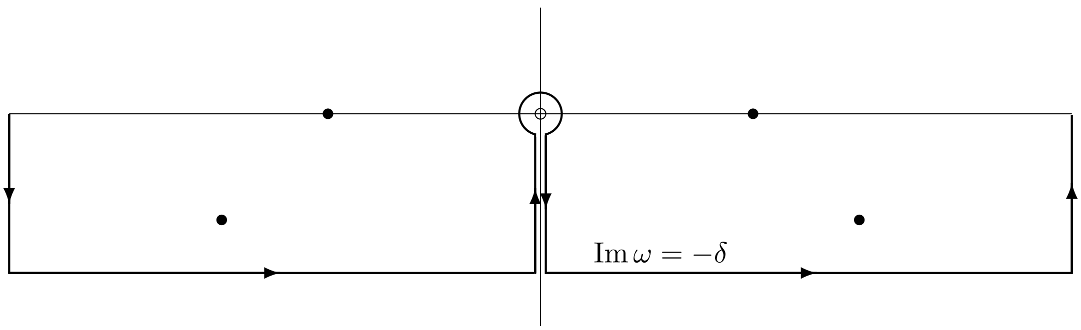


Figure 2: The shifted integration path in the case that there is a branch point at the origin.

complex $\omega$ plane. From the definition of the Fourier transform in equation (13), for any $\omega$ in the upper half plane

$$\overline{f(\omega)} = f(-\overline{\omega}). \tag{51}$$

However, by analytic continuation this relation also holds in $\operatorname{Im}\omega \leq 0$ wherever $f$ is analytic. A consequence of equation (51) is that any poles of $f$ must occur in pairs at $\omega = \omega_n$ and $\omega = -\overline{\omega_n}$, say.

The large-time asymptotic behaviour of $F(t)$ is obtained by moving the path of integration in equation (50) into the lower half of the complex $\omega$ plane and onto the line $\operatorname{Im}\omega = -\delta$, for some $\delta > 0$; see figure 2. It is difficult to make specific statements about what singularities might occur but, in what follows, the asymptotic behaviour that arises from the following singularities of $f$ will be considered, as they are known to occur in certain problems:

(a) simple poles at $\omega = \omega_n \neq 0$, $\delta < \operatorname{Im}\omega_n \leq 0$, and at $\omega = -\overline{\omega_n}$ for $n = 1..M$,

(b) double poles at $\omega = \omega_n \neq 0$, $\operatorname{Im}\omega_n = 0$, and at $\omega = -\omega_n$ for $n = M+1..N$,

(c) a simple pole at the origin,

(d) a double pole at the origin, and

(e) a logarithmic branch point at the origin.

In general

$$F(t) = F_\delta(t) + F_{\text{poles}}(t) + F_{\text{cut}}(t) + F_{\text{inf}}(t) \tag{52}$$

(see Hazard & Loret [10] – although these authors use an equivalent formulation obtained from a Laplace transform in time). Here $F_\delta(t)$ is the contribution from the path along $\operatorname{Im}\omega = -\delta$ (with, if necessary, a break at the branch cut along the imaginary axis), $F_{\text{poles}}(t)$ represents the contributions from any poles crossed when the path of integration is moved, $F_{\text{cut}}(t)$ is the contribution that arises from passing around the branch cut along the negative imaginary axis, and $F_{\text{inf}}(t)$ arises from the closing paths at infinity and is assumed to be negligible (if only the contributions from singularities on the real axis of interest, then the integration path may be deformed so that there are no closing paths at infinity). The contribution from the shifted path,

$$F_\delta(t) = O\left(\mathrm{e}^{-\delta t}\right) \quad \text{as} \quad t \to \infty \tag{53}$$

and from the residue theorem,

$$F_{\text{poles}}(t) = -\mathrm{i}\sum_{n=1}^{N}\left\{\operatorname*{Res}_{\omega=\omega_n} f(\omega)\,\mathrm{e}^{-\mathrm{i}\omega t} + \operatorname*{Res}_{\omega=-\overline{\omega_n}} f(\omega)\,\mathrm{e}^{-\mathrm{i}\omega t} + \operatorname*{Res}_{\omega=0} f(\omega)\,\mathrm{e}^{-\mathrm{i}\omega t}\right\}. \tag{54}$$

The various listed contributions to the large-time asymptotics are as follows:

- Suppose that there is a simple pole at $\omega = \omega_n$ so that

  $$f(\omega) \sim \frac{A_n}{\omega - \omega_n} \quad \text{as} \quad \omega \to \omega_n, \tag{55}$$

  and hence, by virtue of equation (51),

  $$f(\omega) \sim -\frac{\overline{A_n}}{\omega + \overline{\omega_n}} \quad \text{as} \quad \omega \to -\overline{\omega_n}. \tag{56}$$

  A straightforward calculation gives

  $$\operatorname*{Res}_{\omega=\omega_n} f(\omega)\,\mathrm{e}^{-\mathrm{i}\omega t} + \operatorname*{Res}_{\omega=-\overline{\omega_n}} f(\omega)\,\mathrm{e}^{-\mathrm{i}\omega t} = 2\mathrm{i}\operatorname{Im}\left\{A_n\,\mathrm{e}^{-\mathrm{i}\omega_n t}\right\}. \tag{57}$$

  For $\operatorname{Im}\omega_n = 0$ this gives a persistent oscillation in time, while for $\operatorname{Im}\omega_n < 0$, $\operatorname{Re}\omega_n \neq 0$, it gives an oscillation that decays with time. The location of poles in $\operatorname{Im}\omega_n < 0$ and computation of the associated residue can, in principle, be carried out using extensions to complex frequencies of existing solution methods for the frequency-domain problem. An approximation to the location of a pole in the lower half plane can be inferred, quite straightforwardly, from the behaviour of the hydrodynamic coefficients for real frequencies as long as the pole is near the real axis [11].

- Suppose that there is a double pole at $\omega = \omega_n \in \mathbb{R}$ so that

  $$f(\omega) \sim \frac{B_n}{(\omega - \omega_n)^2} \quad \text{as} \quad \omega \to \omega_n, \tag{58}$$

and hence, by virtue of equation (51),

$$f(\omega) \sim \frac{\overline{B_n}}{(\omega + \omega_n)^2} \quad \text{as} \quad \omega \to -\omega_n. \tag{59}$$

In this case the dominant behaviour is

$$\operatorname*{Res}_{\omega=\omega_n} f(\omega)\,\mathrm{e}^{-\mathrm{i}\omega t} + \operatorname*{Res}_{\omega=-\omega_n} f(\omega)\,\mathrm{e}^{-\mathrm{i}\omega t} \sim -2\mathrm{i}t\,\mathrm{Re}\left\{B_n\,\mathrm{e}^{-\mathrm{i}\omega_n t}\right\} \quad \text{as} \quad t \to \infty \tag{60}$$

which is a growing oscillation.

- For a simple pole at the origin, so that

$$f(\omega) \sim \frac{A_0}{\omega} \quad \text{as} \quad \omega \to 0, \tag{61}$$

then

$$\operatorname*{Res}_{\omega=0} f(\omega)\,\mathrm{e}^{-\mathrm{i}\omega t} = A_0. \tag{62}$$

- For a double pole at the origin, so that

$$f(\omega) \sim \frac{B_0}{\omega^2} \quad \text{as} \quad \omega \to 0, \tag{63}$$

then

$$\operatorname*{Res}_{\omega=0} f(\omega)\,\mathrm{e}^{-\mathrm{i}\omega t} \sim -\mathrm{i}B_0 t \quad \text{as} \quad t \to \infty. \tag{64}$$

- To evaluate the contribution from the branch cut it is assumed that, for some non-negative integer $n$,

$$f(\omega) = g(\omega) + C\omega^n \log\omega + o(\omega^n) \quad \text{as} \quad \omega \to 0, \tag{65}$$

where $g(\omega)$ is analytic throughout a neighbourhood of the origin and $C$ is, in general, a complex constant. Immediately to the right of the cut $\omega = u\,\mathrm{e}^{-\mathrm{i}\pi/2}$ and immediately to the left of the cut $\omega = u\,\mathrm{e}^{3\mathrm{i}\pi/2}$, where the real number $u \in [0, \delta]$. Thus

$$\begin{aligned} F_{\mathrm{cut}}(t) &= \frac{1}{2\pi}\left\{\int_\delta^0 [g(-\mathrm{i}u) + C(-\mathrm{i}u)^n(\log u + 3\mathrm{i}\pi/2) + o(u^n)]\,\mathrm{e}^{-ut}(-\mathrm{i}\,\mathrm{d}u)\right. \\ &\qquad \left. + \int_0^\delta [g(-\mathrm{i}u) + C(-\mathrm{i}u)^n(\log u - \mathrm{i}\pi/2) + o(u^n)]\,\mathrm{e}^{-ut}(-\mathrm{i}\,\mathrm{d}u)\right\} \\ &= \frac{1}{2\pi}\left\{C(-\mathrm{i})^n(-2\pi)\int_0^\delta u^n\,\mathrm{e}^{-ut}\,\mathrm{d}u + \text{smaller terms}\right\} \\ &\sim -\frac{C(-\mathrm{i})^n n!}{t^{n+1}} \quad \text{as} \quad t \to \infty, \end{aligned} \tag{66}$$

where the last step follows from Watson's lemma.

## 6 An exact solution with exponential time decay

In general, when the time dependence of the fluid and structure motion in the presence of a free surface is specified explicitly, a velocity potential of the form $\Phi(\mathbf{x}, z, t) = \mathrm{Re}[\phi(\mathbf{x}, z)\, \mathrm{e}^{-\mathrm{i}\omega t}]$, where $\omega$ is real, is sought. This represents time-harmonic motion of the fluid and is coupled to the oscillations of the structure through the body boundary condition (7). However, if the structure is forced to move in a certain fashion, or it is restrained by a spring and damping system, there is no reason why motion with exponential time decay may not exist. As an example, suppose that the structure is two-dimensional and symmetric, intersects the mean, free surface at $x = \pm a$ and is in water of infinite depth. It is assumed to move in the vertical direction under spring and damping constraints but without external forcing. From (11) the equation of motion of the structure is given by

$$M\ddot{X}_3(t) + \gamma_{33}\dot{X}_3(t) + (2\rho g a + \kappa)X_3(t) = -\rho \int_{S_B} \frac{\partial \Phi}{\partial t}(x, z, t)\, n_3 \,\mathrm{d}S. \tag{67}$$

Here $M = \rho V_0$ where $V_0$ is the submerged volume of the structure and $\kappa$ is the spring coefficient. The velocity potential $\Phi$ satisfies (1), (3), (6), (8) and the body boundary condition

$$\frac{\partial \Phi}{\partial n} = \dot{X}_3(t)\, n_3 \quad \text{on} \quad S_B. \tag{68}$$

The structure motion is assumed to have the form $X_3(t) = A\,\mathrm{e}^{-\alpha t}$ and the corresponding fluid velocity potential is given by $\Phi(x, z, t) = -\alpha\, A\, \mathrm{e}^{-\alpha t}\, \phi(x, z)$, where $A$ and $\alpha$ are real constants. The time independent potential $\phi(x, z)$ satisfies (1), (3) and (8). The free surface condition (6) reduces to

$$K\phi + \frac{\partial \phi}{\partial z} = 0, \qquad \text{on } z = 0, \tag{69}$$

where $K = \alpha^2/g$, and the body boundary condition (68) reduces to

$$\frac{\partial \phi}{\partial n} = n_3 \quad \text{on} \quad S_B. \tag{70}$$

There is a different combination of signs in the free surface boundary condition (69) compared to those that occur in the corresponding condition when time-harmonic behaviour is assumed. This ensures that wavelike terms in $\phi$ cannot be supported and so $\phi$ is real. Finally, the equation of motion (67) reduces to

$$\alpha^2 \left[ M + \rho \int_{S_B} \phi(x, z)\, n_3 \,\mathrm{d}S \right] - \alpha\, \gamma_{33} + 2\rho g a + \kappa = 0. \tag{71}$$

Although (71) is not a quadratic equation for $\alpha$ as $\phi$ depends on $\alpha$ through (69), nonetheless real values of $\alpha$ are only possible if

$$\gamma_{33}^2 - 4(2\rho g a + \kappa) \left[ M + \rho \int_{S_B} \phi(x, z)\, n_3 \,\mathrm{d}S \right] \geq 0. \tag{72}$$

Integration of $\nabla.(\phi \nabla \phi)$ over the total fluid region $D$ and an application of the divergence theorem shows that

$$\int_{S_B} \phi(x, z)\, n_3 \,\mathrm{d}S = \iint_D (\nabla \phi)^2 \,\mathrm{d}V + K \int_{|x|>a} \phi^2(x, 0)\, \mathrm{d}x > 0. \tag{73}$$

Thus, no real values of $\alpha$ which satisfy (71) are possible unless either the damping coefficient $\gamma_{33}$ is non-zero and/or the spring coefficient $\kappa$ is negative, and both have sufficient magnitude. In particular, motion of the structure which decays exponentially in time with no oscillations, is not possible if the structure is allowed to move freely.

A source-like solution of (1), (3), (8) and (69) is given by

$$G(x,z) = -\int_0^\infty \frac{\mathrm{e}^{\mu z}\,\cos\mu x}{\mu+K}\,\mathrm{d}\mu = -\,\mathrm{Re}\left[\mathrm{e}^{-K(z+\mathrm{i}x)}\,E_1(-K(z+\mathrm{i}x))\right], \tag{74}$$

and symmetric, wave-free potentials are given by

$$\phi_n^s(r,\theta) = \frac{K}{2n-1}\frac{\cos(2n-1)\theta}{r^{2n-1}} - \frac{\cos 2n\theta}{r^{2n}}, \quad n = 1,2,\ldots \tag{75}$$

where $x = r\sin\theta$ and $z = -r\cos\theta$. They may be obtained by replacing $K$ by $-K$ in the corresponding frequency domain potentials derived by Ursell [16] and noting that as there is no singularity in the integrand in (74), there are no waves generated by the source potential. In order for $\phi$ to satisfy (3) and (8)

$$\phi \sim \mu\,\phi_1^s(r,\theta) = \mu\left(\frac{K\cos\theta}{r} - \frac{\cos 2\theta}{r^2}\right), \quad \text{as} \quad r\to\infty, \tag{76}$$

where the dipole coefficient $\mu$ is real. Application of Green's theorem to $\phi$ and the 'growing' potential $\phi_0^s = Kz - 1$ gives, after some manipulation,

$$\alpha^2\left[M + \rho\int_{S_B}\phi(x,z)\,n_3\,\mathrm{d}S\right] + 2\rho g a - \mu\rho\pi\frac{\alpha^4}{g} = 0. \tag{77}$$

A comparison of (77) and (73) shows that $\mu > 0$ and a comparison of (77) and (71) shows that it satisfies

$$\mu\rho\pi\frac{\alpha^4}{g} = \alpha\,\gamma_{33} - \kappa. \tag{78}$$

The source potential given in (74) has an asymptotic expansion in terms of wave-free potentials which is not the case for the corresponding time-harmonic source as this contains wave-like terms. Thus in principle, the potential associated with a vertically moving structure such as a half-immersed circular cylinder, could be sought as an expansion in terms of wave-free potentials in an analogous fashion to the work done by Ursell [16]. The mooring constraints needed to support this motion would then be determined from (78). However, as the intention here is simply to establish that such motion of a structure may exist, the inverse procedure described in [17] is used to construct the structure from a given potential. The potential is specified as a multiple of the source, namely

$$\phi = -\frac{1}{K}G(x,z) = -\frac{1}{K}\,\mathrm{Re}\left[\mathrm{e}^{-K(z+\mathrm{i}x)}\,E_1(-K(z+\mathrm{i}x))\right], \tag{79}$$

where the sign in $\phi$ is chosen to ensure that the dipole coefficient is positive and the factor of $1/K$, to ensure that $\phi$ has the correct dimensions. From (70) the corresponding structure is a streamline of the flow associated with the potential $\phi(x,z) - z$, which has the property that it removes the singularity from the fluid. The stream function associated with the source potential is given by

$$H(x,z) = \int_0^\infty \frac{\mathrm{e}^{\mu z}\,\sin\mu x}{\mu + K}\,\mathrm{d}\mu = \mathrm{Im}\left[\mathrm{e}^{-K(z+\mathrm{i}x)}\,E_1(-K(z+\mathrm{i}x))\right], \tag{80}$$

and so the structure surface $S_B$ must be a suitable streamline of the function $\tilde{\psi} = -H(x,z)/K + x$. Figure 3 illustrates the streamline pattern for $K\tilde{\psi}$ , where distances have been nondimensionalised with the parameter $K$. Clearly, there is a streamline which divides the streamlines that go into the

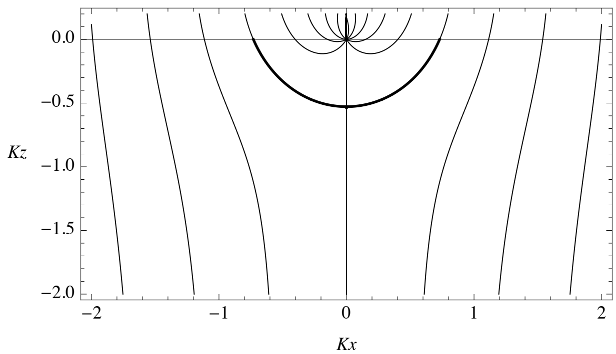


Figure 3: Streamlines of $K\tilde{\psi}$

source point from those that extend to infinity. This streamline (shown as a thick line in figure 3) removes the source point from the fluid and may be interpreted as a moving structure.

# 7 Large-time asymptotics in the water-wave problem

## 7.1 Limiting oscillatory behaviour

The residues at any simple poles of a frequency-domain potential that occur on the real axis in the complex $\omega$ plane (other than at $\omega = 0$) contribute time-harmonic behaviour to the time-domain solution – see equation (57) – and such oscillations will dominate the large-time behaviour. (For simplicity, in this subsection, it is assumed that there are no double poles that correspond to growth of the solution with time.) Real poles at $\omega = \omega_0 \neq 0$ in the frequency-domain potential for a freely-floating structure may arise in the following ways.

(a) When the incident wave has a component that corresponds to a persistent oscillation with frequency $\omega_0 \in \mathbb{R}$, then in general the exciting force $\xi_p(\omega)$ will have poles at $\omega = \pm\omega_0$.

(b) When the external force corresponds to a persistent oscillation with frequency $\omega_0 \in \mathbb{R}$, then $f_p(\omega)$ will have poles at $\omega = \pm\omega_0$.

(c) When the coefficient matrix for the system of equations (40) is singular at $\omega = \omega_0 \in \mathbb{R}$ (such a singularity corresponds to a so-called "motion trapped mode" which is a free coupled oscillation of the fluid and structure [17, 18]), then any of the $v_p(\omega)$ may have poles at $\omega = \pm\omega_0$.

In both of cases (a) and (b), the corresponding large-time behaviour is oscillatory with the same frequency as the forcing and will not depend upon the initial conditions provided that the structure does not support trapped modes (the principle of limiting amplitude [19]). When a motion trapped mode is supported by the structure the large-time behaviour may be influenced by all terms on the right-hand side of equations (40), including those that involve the initial conditions. (For example, the amplitude of the trapped mode is determined by the initial conditions.) In the absence of trapped modes, that part of the time-domain solution that arises from the initial conditions represents a transient phenomenon that decays to zero as $t \to \infty$.

## 7.2 Vertical motion of a structure released from rest

Consider a structure that has the symmetry necessary to ensure that it will move only vertically and that is subject to no external forces, including those from moorings. The structure is released from rest in the absence of incident waves so that the equation of motion (40) and the relation (20) yield a frequency-domain displacement

$$x_3(\omega) = \frac{-\mathrm{i}\omega[M + \mu_{33}(\omega) + \mathrm{i}\nu_{33}(\omega)]X_3(0)}{\rho g W - \omega^2[M + \mu_{33}(\omega) + \mathrm{i}\nu_{33}(\omega)]}. \tag{81}$$

Here $M$ is the mass of the structure, the term $\rho g W \equiv c_{33}$ is the hydrostatic spring coefficient, and $W$ is the water-plane area which has dimension of length in two dimensions, and of length squared in three dimensions. It will be assumed that the structure does not support motion trapped modes, so that $x_3(\omega)$ has no poles for any real $\omega \neq 0$, and attention is restricted to the contributions to the long-time asymptotics that arise from the origin in the frequency domain. There will be poles of $x_3(\omega)$ in the lower half of the complex $\omega$ plane that will contribute exponentially decaying oscillations to the long-time asymptotics but, provided they are non-zero, the algebraic contributions that arise from the origin will provide the ultimate decay with time (although such a decay is unlikely to be observed in practice [20]). The asymptotics of the hydrodynamic coefficients as $\omega \to 0$ for real $\omega$ are readily available, and arguments based on integral equations and the low-frequency behaviour of the relevant Green's functions can be used to justify the use of these asymptotics in the complex $\omega$ plane.

Four cases may be distinguished as follows:

1. For a two-dimensional surface-piercing structure in deep water, the complex force coefficient $\varpi_{33}(\omega)$ has a logarithmic branch point at the origin [7]; more specifically, with $W = 2a$ say,

$$\varpi_{33}(\omega) \equiv \mu_{33}(\omega) + \mathrm{i}\nu_{33}(\omega) \sim -\frac{8\rho a^2}{\pi}\log\omega + D \quad \text{as} \quad \omega \to 0 \tag{82}$$

where $D$ is a complex constant [21, section 5.1]. Thus

$$x_3(\omega) \sim -\frac{\mathrm{i}\omega}{2\rho g a}\left[M - \frac{8\rho a^2}{\pi}\log\omega + D\right]X_3(0) \quad \text{as} \quad \omega \to 0 \tag{83}$$

and then equation (66) yields

$$X_3(t) \sim -\frac{4aX_3(0)}{\pi g t^2} \quad \text{as} \quad t \to \infty, \tag{84}$$

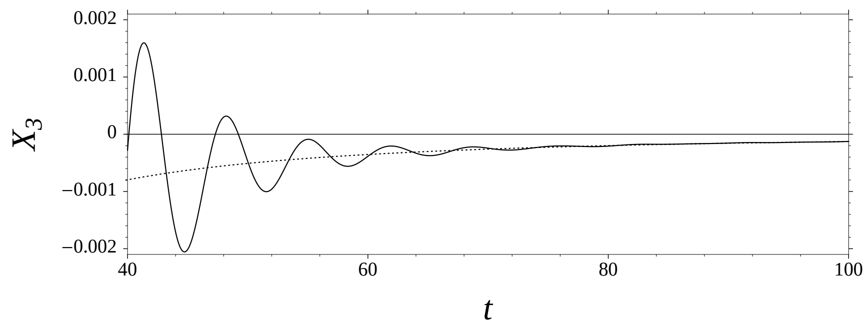


Figure 4: The displacement $X_3(t)$ (———) for a half-immersed circular cylinder that is released from rest with $X_3(0) = 1$. The dotted line indicates the large-time asymptotics given by equation (84).

in agreement with the calculation of Ursell [7, equation 4.14] for a half-immersed circular cylinder. Computations of the decaying oscillations that dominate all but the later stages of the motion have been made by a number of authors (e.g. [20, 22]). Here, the above large-time asymptotics are compared in figure 4 with a time-domain solution computed using the reduction of equations (43) for mode 3, confirming the conclusion in [20] that the ultimate algebraic decay is essentially negligible compared to the initial displacement which is chosen as $X_3(0) = 1$ (here, and in the later computations, units are chosen so that $\rho$, $g$, and the radius of the cylinder $a$, all have value one).

2. For a three-dimensional surface-piercing structure with a vertical axis of symmetry in deep water, the complex force coefficient $\varpi_{33}(\omega)$ again has a branch point at the origin and, with $W = \pi a^2$ say,

$$\varpi_{33}(\omega) \sim D - \omega^2 \left( \frac{\rho \pi a^4}{g} \log \omega + E \right) \quad \text{as} \quad \omega \to 0, \tag{85}$$

where $D$ and $E$ are complex constants [21, section 7.1]. Thus

$$x_3(\omega) \sim -\frac{\mathrm{i}\omega}{\rho g \pi a^2} \left[ M + D - \omega^2 \left( \frac{\rho \pi a^4}{g} \log \omega + E \right) \right] X_3(0) \quad \text{as} \quad \omega \to 0 \tag{86}$$

and then equation (66) yields

$$X_3(t) \sim \frac{6a^2 X_3(0)}{g^2 t^4} \quad \text{as} \quad t \to \infty, \tag{87}$$

in agreement with the calculation of Kotik & Lurye [8, equation 3.29].

3. For a two-dimensional surface-piercing structure in water of finite depth, the complex force coefficient $\varpi_{33}(\omega)$ has a simple pole at the origin [23], but no branch point, so that $x_3(\omega)$ is analytic at the origin. Consequently, there is no ultimate algebraic decay of the structure when released from rest and, in general, the structure oscillates for all time with a decaying amplitude.

4. For a three-dimensional surface-piercing structure with a vertical axis of symmetry in water of finite depth $h$, the complex force coefficient $\varpi_{33}(\omega)$ has a branch point at the origin and, with $W = \pi a^2$ say,

$$\varpi_{33}(\omega) \sim -\frac{\rho\pi a^4}{2h}\log\omega + D \quad \text{as} \quad \omega \to 0, \tag{88}$$

where $D$ is a complex constant [21, section 10.1]. Thus

$$x_3(\omega) \sim -\frac{\mathrm{i}\omega}{\rho g\pi a^2}\left[M - \frac{\rho\pi a^4}{2h}\log\omega + D\right] X_3(0) \quad \text{as} \quad \omega \to 0 \tag{89}$$

and then equation (66) yields

$$X_3(t) \sim -\frac{a^2 X_3(0)}{2ght^2} \quad \text{as} \quad t \to \infty; \tag{90}$$

this result is similar to that for a two-dimensional structure in deep water, given in equation (84), and appears to be new.

### 7.3 Motion of an asymmetric structure in the absence of incident waves

Consider a structure that is free to move in surge ($p = 1$) and heave ($p = 3$), but is not subject to incident waves or external forcing and is not restrained by any moorings. Initial conditions are chosen such that $X_1(0) = V_1(0) = 0$ and motion is initiated through non-zero values for $X_3(0)$ and/or $V_3(0)$. As $\omega \to 0$, the added mass $\mu_{11}(\omega)$ tends to a non-zero constant and $\nu_{11}(\omega) \to 0$ and then, after solution of the equations of motion (40) and use of the relation (20), it is fairly straightforward to obtain

$$x_1(\omega) \sim -\frac{\mu_{13}(\infty)V_3(0)}{\omega^2[M + \mu_{11}(0)]} \quad \text{as} \quad \omega \to 0. \tag{91}$$

when $V_3(0) \neq 0$. Under the assumption that $\mu_{13}(\infty) \neq 0$, equation (64) gives a dominant behaviour

$$X_1(t) \sim \frac{\mu_{13}(\infty)V_3(0)\,t}{M + \mu_{11}(0)} \quad \text{as} \quad t \to \infty. \tag{92}$$

(The asymptotics of $\mu_{13}(\omega)$ at low and high frequency do not appear to have been investigated. However, numerical computations by a two-dimensional boundary-element code for an inclined, submerged ellipse suggest that both $\mu_{13}(0) \neq 0$ and $\mu_{13}(\infty) \neq 0$.) In the case that $V_3(0) = 0$ and $\mu_{13}(0) \neq 0$, then

$$x_1(\omega) \sim \frac{\mathrm{i}\mu_{13}(0)X_3(0)}{\omega[M + \mu_{11}(0)]} \quad \text{as} \quad \omega \to 0. \tag{93}$$

and equation (62) gives

$$X_1(t) \sim \frac{\mu_{13}(0)X_3(0)}{M + \mu_{11}(0)} \quad \text{as} \quad t \to \infty. \tag{94}$$

The last result indicates that for an asymmetric structure that is displaced vertically and then released from rest, there is a net horizontal translation proportional to the initial displacement. On the other hand, when a vertical velocity is imposed this results in a horizontal translation that continues to increase with time and the assumptions of the linearised theory used here are violated (a revised theory might possibly be constructed that linearises about the translating state, provided such translations are a property of the full nonlinear problem). Similar horizontal translations induced by asymmetry are discussed by Porter & Evans [24] in the context of wave trapping by floating circular cylinders.

### 7.4 Motion due to incident waves

Consider a structure that has the symmetry necessary to ensure that the motions in different modes are independent and that is subject to no external forces, including those from moorings. The structure is assumed to be initially at rest in a state of equilibrium and subject to incident waves so that the equations of motion (40) and the relation (20) yield a frequency-domain displacement for mode $p$ of

$$x_p(\omega) = \frac{\widetilde{\Phi}_{\mathrm{S},p} + \xi_p(\omega)}{c_{pp} - \omega^2[M_{pp} + \mu_{pp}(\omega) + \mathrm{i}\nu_{pp}(\omega)]}. \tag{95}$$

For vertical motion, $p = 3$, of a surface-piercing structure then $c_{pp} = \rho g W \neq 0$ and as a consequence $x_3(\omega)$ is not singular at $\omega = 0$ so that there are no corresponding contributions to the long-time asymptotics of the sort listed in section 5; see [21] for details of the relevant low-frequency asymptotics of the added mass and damping coefficients. The large-time behaviour is a combination of decaying oscillations arising from any poles of $x_3(\omega)$ in the lower half of the complex $\omega$ plane.

For horizontal motions, $p = 1$, the situation is more interesting as $c_{11} = 0$ so that

$$x_1(\omega) = \frac{\widetilde{\Phi}_{\mathrm{S},1} + \xi_1(\omega)}{-\omega^2[M + \mu_{11}(\omega) + \mathrm{i}\nu_{11}(\omega)]}, \tag{96}$$

where $M$ is the mass of the structure. Singular behaviour at $\omega = 0$ is now possible and, in general, $x_1(\omega)$ has a double pole at the origin. From equation (64), the large time asymptotics are such that $|X_1(t)|$ increases linearly with time at a rate that is proportional to $\widetilde{\Phi}_{\mathrm{S},1} + \xi_1(0)$. Although this is formally a solution to the problem, the behaviour contradicts the main assumption made in the linearisation of the problem as long as $\widetilde{\Phi}_{\mathrm{S},1} + \xi_1(0) \neq 0$. Further investigation is needed to determine whether or not this is possible in circumstances of practical interest.

### 7.5 Horizontal motion due to an external force

The structure is assumed to be initially at rest in a state of equilibrium and subject to an external force so that the equations of motion (40) and the relation (20) yield a frequency-domain displacement for mode 1 of

$$x_1(\omega) = \frac{f_1(\omega)}{-\omega^2[M + \mu_{11}(\omega) + \mathrm{i}\nu_{11}(\omega)]}. \tag{97}$$

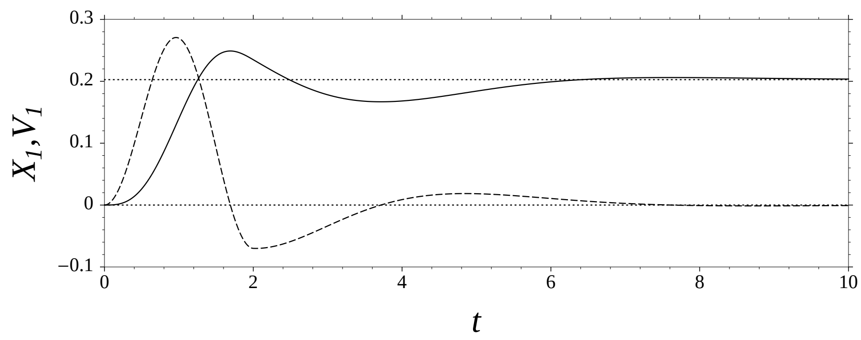


Figure 5: The displacement $X_1(t)$ (———) and the velocity $V_1(t)$ (– – – –) for a half-immersed circular cylinder subject to the force in equation (98). The dotted lines indicate the predicted large-time asymptotics.

For a force

$$F_1(t) = \begin{cases} F_0 \sin \omega_0 t, & 0 \le t \le 2\pi/\omega_0, \\ 0, & t > 2\pi/\omega_0, \end{cases} \tag{98}$$

the Fourier transform

$$f_1(\omega) = \frac{\omega_0 F_0 \left[ \mathrm{e}^{2\pi \mathrm{i}\omega/\omega_0} - 1 \right]}{\omega^2 - \omega_0^2} \sim -\frac{2\pi \mathrm{i} \omega F_0}{\omega_0^2} \quad \text{as} \quad \omega \to 0. \tag{99}$$

It follows that

$$x_1(\omega) \sim \frac{2\pi \mathrm{i} F_0}{\omega \omega_0^2 [M + \mu_{11}(0)]} \quad \text{as} \quad \omega \to 0 \tag{100}$$

and hence, from equation (62),

$$X_1(t) \sim \frac{2\pi F_0}{\omega_0^2 [M + \mu_{11}(0)]} \quad \text{as} \quad t \to \infty \tag{101}$$

so that there is a net displacement of the structure proportional to the magnitude of the force. This result is illustrated for a half-immersed circular cylinder in deep water in figure 5, where a time-domain solution computed using the reduction of equations (43) for mode 1 is compared with the above large-time asymptotics (with $F_0 = 1$ and $\omega_0 = \pi$).

It is straightforward to choose a force that imparts a non-zero velocity on the structure as $t \to \infty$. For example, with

$$F_1(t) = \begin{cases} F_0(1 - t/T), & 0 \le t \le T, \\ 0, & t > T, \end{cases} \tag{102}$$

the Fourier transform

$$f_1(\omega) = \frac{(1 + \mathrm{i}\omega T - \mathrm{e}^{\mathrm{i}\omega T}) F_0}{\omega^2 T} \sim \tfrac{1}{6} F_0 T \, (3 + \mathrm{i}\omega T) \quad \text{as} \quad \omega \to 0 \tag{103}$$

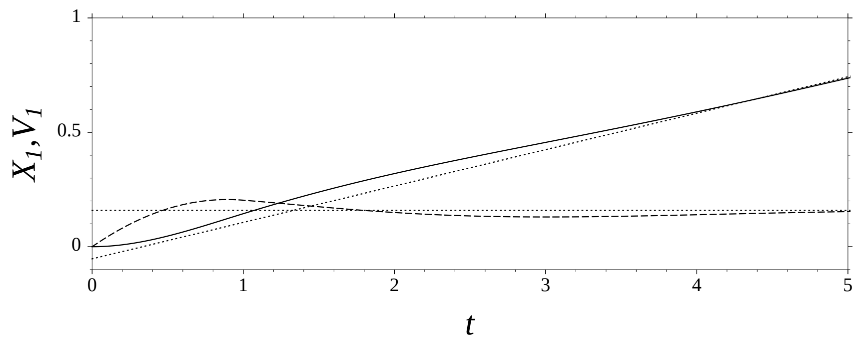


Figure 6: The displacement $X_1(t)$ (———) and the velocity $V_1(t)$ (– – – –) for a half-immersed circular cylinder subject to the force in equation (102). The dotted lines indicate the predicted large-time asymptotics.

which gives

$$x_1(\omega) \sim -\frac{F_0 T(3 + \mathrm{i}\omega T)}{6\omega^2[M + \mu_{11}(0)]} \quad \text{as} \quad \omega \to 0 \tag{104}$$

and so

$$X_1(t) \sim \frac{F_0 T(3t - T)}{6[M + \mu_{11}(0)]} \quad \text{as} \quad t \to \infty. \tag{105}$$

This result is illustrated in figure 6, again for the motion of a half-immersed circular cylinder in deep water computed using the reduction of equations (43) for mode 1 (with $F_0 = 1$ and $T = 1$).

# 8 Conclusion

In this work the time-domain behaviour of the motion of a floating structure has been investigated using linear theory and two new phenomena have been identified. First of all, it has been shown that solutions of the water-wave problem exist that have a purely exponential, non-oscillatory, decay with time. Secondly, it has been shown that there are solutions that at large times involve a steady translation of the structure, and hence violate the assumptions made in the linearisation of the problem. Although it has been shown that such motions might, in principle, be excited by an incident wave field, it is desirable to shed further light on this phenomenon by making some specific calculations, and this will be the subject of future work.

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