

Water waves over arrays of horizontal cylinders: band gaps and Bragg resonance

C. M. LINTON

Department of Mathematical Sciences, Loughborough University, Leicestershire,
LE11 3TU, UK

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The existence of a band-gap structure associated with water waves propagating over infinite periodic arrays of submerged horizontal circular cylinders in deep water is established. The cases of waves propagating at right angles to the cylinder axes and at an oblique angle are both considered. Calculations for large finite arrays are also presented which show the effect of an incident wave having a frequency within a band gap—with the amount of energy transmitted across the array tending to zero as the size of the array is increased. The location of the band gaps is not as predicted by Bragg’s law, but we show that an approximate determination of their position can be made very simply if the phase of the transmission coefficient for a single cylinder is known.

1. Introduction

When a plane wave is incident upon a periodic structure (periodic in the x -direction with period d , say), a resonance can occur when the component of the wavenumber in the x -direction is an integer multiple of π/d . This is known as Bragg’s law following William Lawrence Bragg’s discovery of the phenomenon in the context of X-ray diffraction in 1912 (Hunter (2004), Chapter 2). If we denote the angle that the wave makes with the x -axis by φ then this is equivalent to the condition $2d \cos \varphi = n\lambda$, where λ is the incident wavelength. For incident waves in the x -direction this leads to the idea that resonances may occur when the wavelength is twice the period.

In the context of water waves (and for the purposes of this paper we exclude cases where the depth variation can be factored out leading to an equivalent acoustics problem) Bragg resonance has been used to help explain the formation of longshore sandbars which can often be found on gently sloping beaches; see, for example, Mei *et al.* (2005), Chapter 7. The particular theoretical problem that has been the focus of most attention is the scattering of surface waves by a finite patch of sinusoidally varying topography in an otherwise flat ocean; see Davies (1982), Davies & Heathershaw (1984), Mei (1985), Mei *et al.* (1988), Chamberlain & Porter (1995). The inclusion of an elastic plate on the free surface (modelling an ice sheet) has also been considered (Bennetts *et al.* (2009)). More recently, the effect of Bragg resonance on the efficiency of certain types of wave-energy device has been investigated by Garnaud & Mei (2010). The analysis in these papers is predicated on the scatterers (bed ripples in the former case and floating buoys in the latter) being small, which then allows for some form of approximation to be made. By contrast, in this article we consider a periodic array made up of structures whose dimensions are of comparable size to the wavelength.

Specifically, we consider wave interaction with a periodic array of horizontal circular cylinders of radius a submerged below the free surface in water of infinite depth. Everything that follows could easily be adapted to the case of constant finite depth. The undisturbed free surface is $z = 0$ and the cylinder centres are at $x = jd$, $z = -f$, where

$j \in \mathbb{Z}$ in the case of an infinite array, or $j = 0, \dots, M-1$ if there are M cylinders. Polar coordinates centred at $x = jd$, $z = -f$ are (r_j, θ_j) with θ_j measured from the downward vertical so that

$$r_j \sin \theta_j = x - jd, \quad r_j \cos \theta_j = -z - f. \quad (1.1)$$

We will write (r, θ) for (r_0, θ_0) where convenient. The analysis is based on linear water wave theory in which the fluid is assumed to be inviscid and incompressible and its motion assumed to be irrotational. A time factor of $\exp(-i\omega t)$ is suppressed throughout. We assume an $\exp(i\ell y)$ ($\ell \geq 0$) dependence for the velocity potential so that in the region $z < 0$, $r_j > a$ we have to solve the two-dimensional modified Helmholtz equation $\nabla_{xz}^2 \phi - \ell^2 \phi = 0$. The free-surface boundary condition is $\phi_z = K\phi$ on $z = 0$ ($K = \omega^2/g$, g being the acceleration due to gravity), and we require decay as $z \rightarrow -\infty$. The cylinders are assumed to be rigid so that $\partial\phi/\partial r_j = 0$ on $r_j = a$.

The primary aim of this paper is to illustrate the link between resonant behaviour associated with scattering by finite periodic structures and the concept of stop bands in wave propagation through an infinite periodic array. The existence of ranges of frequencies for which wave propagation through a periodic array is not possible is well known, and of great practical importance, in a host of physical contexts. In water waves the most thorough investigations, in which the full linear problem including evanescent interactions is included, appear to be those by Chou (1998), McIver (2000) and Porter & Porter (2003). There are many other studies, e.g. An & Ye (2004), Chen *et al.* (2004), Yang *et al.* (2006) in which the evanescent contributions are ignored. If an infinite periodic array has a stop band then one would expect that this would lead to enhanced reflection when a plane wave is incident on a finite array at a frequency within such a stop band. Moreover, under such circumstances one would expect that as the size of the finite array increases, the amount of energy transmitted across the array should tend to zero.

If one assumes that the scatterers are widely spaced so that the interaction between them is governed solely by the modes propagating along the surface and not the evanescent field, then the problem is greatly simplified and the overall effect of an array of M cylinders can be determined in terms of the scattering characteristics of a single cylinder using a transfer matrix approach. This is described briefly in §2 and it leads to a very simple condition which can be used to determine the stop and pass bands for an infinite array. One of the motivations for considering submerged circular cylinders in this context is that in the case $\ell = 0$ (in deep water) the wide-spacing argument suggests that stop bands do not exist, essentially due to the fact that the reflection coefficient for a single cylinder is identically zero (Dean (1948)). The full linear theory, based on multipole expansions (first used for a single cylinder by Ursell (1950)), is described in §3 for the case $\ell = 0$ and in §4 for $\ell > 0$. Numerical results are presented in §5, showing that stop bands do exist for all $\ell \geq 0$, leading to the possibility of enhanced reflection for large finite arrays when an incident wave has a frequency which lies within one of these bands.

2. Finite array of widely-spaced cylinders

The analysis in this section is described in terms of infinite depth, but the results are equally valid for the case of constant finite depth with the appropriate change in wavenumber. We consider an array of scatterers at $x = jd$, $j = 0, \dots, M-1$. They need not be rigid circular cylinders, the only requirement for what follows is that they be identical. The results are equivalent (though expressed in a somewhat simpler form) to those given in Evans (1990); see also Linton & McIver (2001), §6.3.

First we consider a single scatterer at $x = 0$ and assume that for a wave with x -

variation $\exp(i\alpha x)$, $\alpha = (K^2 - \ell^2)^{1/2} > 0$, incident from $x = -\infty$ we have reflection and transmission coefficients R and T , respectively. Similarly, for a wave $\exp(-i\alpha x)$ incident from $x = \infty$ we have reflection and transmission coefficients r and t , respectively. From standard linear water wave theory we have (Linton & McIver (2001), §1.4)

$$t = T, \quad |r| = |R|, \quad |R|^2 + |T|^2 = |r|^2 + |t|^2 = 1, \quad R\bar{t} + \bar{r}T = 0. \quad (2.1)$$

If the scatterer is at $x = jd$, then we will label the reflection and transmission coefficients as R_j , r_j and T_j , t_j , respectively, and we have

$$T_j = t_j = T, \quad R_j = R e^{2ij\alpha d}, \quad r_j = r e^{-2ij\alpha d}. \quad (2.2)$$

For a single scatterer at $x = jd$ we define the transfer matrix S_j by

$$\begin{pmatrix} A_{j+1} \\ B_{j+1} \end{pmatrix} = S_j \begin{pmatrix} A_j \\ B_j \end{pmatrix}, \quad (2.3)$$

which relates the amplitudes of the waves to the right of the scatterer (A_{j+1} propagating to the right and B_{j+1} to the left) to those on the left (A_j propagating to the right and B_j to the left). The definitions of the reflection and transmission coefficients imply that

$$\begin{pmatrix} T_j & r_j \\ 0 & 1 \end{pmatrix} = S_j \begin{pmatrix} 1 & 0 \\ R_j & t_j \end{pmatrix} \quad (2.4)$$

from which

$$S_j = \frac{1}{t_j} \begin{pmatrix} T_j t_j - R_j r_j & r_j \\ -R_j & 1 \end{pmatrix} = \begin{pmatrix} 1/\bar{t} & (r/t) e^{-2ij\alpha d} \\ (r/t) e^{2ij\alpha d} & 1/t \end{pmatrix} \quad (2.5)$$

on use of (2.1) and (2.2). If we write $r = \rho \exp(2i\delta_r)$ and $t = \tau \exp(2i\delta_t)$ we obtain

$$S_j = \begin{pmatrix} e^{i\alpha_j} & 0 \\ 0 & e^{-i\alpha_j} \end{pmatrix} \begin{pmatrix} 1/\tau & \rho/\tau \\ \rho/\tau & 1/\tau \end{pmatrix} \begin{pmatrix} e^{i\beta_j} & 0 \\ 0 & e^{-i\beta_j} \end{pmatrix} = A_j R B_j, \quad (2.6)$$

say, where

$$\alpha_j = \delta_r - j\alpha d, \quad \beta_j = 2\delta_t - \delta_r + j\alpha d. \quad (2.7)$$

For an array of scatterers at $x = jd$, $j = 0, \dots, M-1$ and an incident wave from the left with x -dependence $\exp(i\alpha x)$, we denote the overall reflection and transmission coefficients by R_M and T_M , respectively. If we treat the scatterers as widely separated so that multiple scattering effects are limited to the propagating waves, then we have

$$\begin{pmatrix} T_M \\ 0 \end{pmatrix} = A_{M-1} R B_{M-1} A_{M-2} R B_{M-2} \dots A_1 R B_1 A_0 R B_0 \begin{pmatrix} 1 \\ R_M \end{pmatrix}. \quad (2.8)$$

Let $\chi = \beta_{j+1} + \alpha_j = 2\delta_t + \alpha d$, which is independent of j . Then

$$B_{j+1} A_j R = \begin{pmatrix} e^{i\chi/\tau} & \rho e^{i\chi/\tau} \\ \rho e^{-i\chi/\tau} & e^{-i\chi/\tau} \end{pmatrix} = W, \quad \text{say}, \quad (2.9)$$

and

$$\begin{pmatrix} T_M \\ 0 \end{pmatrix} = B_M^{-1} W^M B_0 \begin{pmatrix} 1 \\ R_M \end{pmatrix}. \quad (2.10)$$

It is clear that $\det W = 1$ and $\text{trace } W = (2/\tau) \cos \chi$ which is real. Hence the eigenvalues of W are

$$\lambda_1 = e^{iq} \quad \text{and} \quad \lambda_2 = e^{-iq}, \quad (2.11)$$

where $\cos q = (1/\tau) \cos \chi$ and q will be real or complex depending on whether $\tau > |\cos \chi|$

or $\tau < |\cos \chi|$. We then have (Markoš & Soukoulis (2008), §1.5.2)

$$\mathbb{W}^n = \begin{pmatrix} (1/\tau)e^{ix}U_{n-1} - U_{n-2} & (\rho/\tau)e^{ix}U_{n-1} \\ (\rho/\tau)e^{-ix}U_{n-1} & (1/\tau)e^{-ix}U_{n-1} - U_{n-2} \end{pmatrix}, \quad (2.12)$$

where

$$U_n \equiv U_n(\cos q) = \frac{\sin(n+1)q}{\sin q} \quad (2.13)$$

is a Chebyshev polynomial of the second kind. Thus

$$\mathbb{B}_M^{-1}\mathbb{W}^M\mathbb{B}_0 = \begin{pmatrix} [(1/\tau)e^{ix}U_{M-1} - U_{M-2}]e^{i(\beta_0 - \beta_M)} & (\rho/\tau)e^{ix}U_{M-1}e^{-i(\beta_0 + \beta_M)} \\ (\rho/\tau)e^{-ix}U_{M-1}e^{i(\beta_0 + \beta_M)} & [(1/\tau)e^{-ix}U_{M-1} - U_{M-2}]e^{i(\beta_M - \beta_0)} \end{pmatrix}. \quad (2.14)$$

Substituting this into (2.10), noting that $\mathbb{B}_M^{-1}\mathbb{W}^M\mathbb{B}_0$ has determinant one, yields

$$T_M = \frac{\tau e^{-iM\alpha d}}{e^{-ix}U_{M-1} - \tau U_{M-2}}. \quad (2.15)$$

A straightforward calculation then shows that

$$|T_M|^2 = \frac{\tau^2}{\tau^2 + \rho^2 U_{M-1}^2}. \quad (2.16)$$

If q is real, then clearly $|T_M|^2$ oscillates as M varies. However, if q is complex, which happens if

$$\tau < |\cos \chi|, \quad (2.17)$$

then $|T_M| \rightarrow 0$ as $M \rightarrow \infty$. Hence, in the context of water waves, (2.17) is the condition for a stop band in an infinite array of widely-spaced scatterers (a result which appears to have been given first by McIver (2000)). Some general conclusions follow from (2.17). The smaller the value of τ (i.e. the stronger the individual elements of the array are at reflecting waves) the larger the band gaps are likely to be, and one would expect the band gaps to be centred around the points $\chi = n\pi$. If the phase of the transmission coefficient is small then these points are approximately $\alpha d = n\pi$, which is Bragg's law. A classic example of a scatterer for which δ_t is not negligible even though it is very poor at reflecting waves is a submerged horizontal circular cylinder in deep water, for which $\rho \equiv 0$ if $\ell = 0$, and the rest of the paper deals with arrays of such scatterers.

It is possible to extend the transfer matrix approach to include evanescent wave interactions (Devillard *et al.* (1988)) and this has been done in the context of periodic bed topographies in Porter & Porter (2003). Here we take a different approach to solving the full linear problem.

3. Full linear theory: waves perpendicular to the cylinders

When $\ell = 0$ the modified Helmholtz equation reduces to Laplace's equation. In this case, the ingredients for treating an array of submerged circular cylinders using multipole expansions can be found in O'Leary (1985) and Linton & McIver (2001), §3.2. Multipoles singular at $r_j = 0$ and symmetric in θ_j are (Linton & McIver (2001), eqn B.17)

$$\phi_n^j = \frac{\cos n\theta_j}{r_j^n} + \frac{(-1)^n}{(n-1)!} \int_0^\infty \frac{\mu + K}{\mu - K} \mu^{n-1} e^{\mu(z-f)} \cos \mu(x - jd) d\mu, \quad n \geq 1 \quad (3.1)$$

and multipoles singular at $r_j = 0$ and antisymmetric in θ_j are (Linton & McIver (2001), eqn B.26)

$$\psi_n^j = \frac{\sin n\theta_j}{r_j^n} + \frac{(-1)^{n+1}}{(n-1)!} \int_0^\infty \frac{\mu + K}{\mu - K} \mu^{n-1} e^{\mu(z-f)} \sin \mu(x - jd) d\mu, \quad n \geq 1. \quad (3.2)$$

For $z > f$ we also have (Linton & McIver (2001), eqn B.3)

$$\phi_n^j = \frac{2(-1)^n}{(n-1)!} \int_0^\infty \frac{\mu^{n-1} e^{-\mu f}}{\mu - K} (\mu \cosh \mu z + K \sinh \mu z) \cos \mu(x - jd) d\mu, \quad (3.3)$$

and for ψ_n^j one simply replaces \cos by $-\sin$ in this expression.

We can develop series expansions for these multipoles about the point $x = kd$, $z = -f$. Thus, for all j, k and for $r_k < 2f$,

$$\phi_n^j = \frac{\cos n\theta_j}{r_j^n} + \sum_{m=0}^{\infty} P_{mn}^{jk} r_k^m \cos m\theta_k - \sum_{m=1}^{\infty} Q_{mn}^{jk} r_k^m \sin m\theta_k \quad (3.4)$$

$$\psi_n^j = \frac{\sin n\theta_j}{r_j^n} + \sum_{m=0}^{\infty} Q_{mn}^{jk} r_k^m \cos m\theta_k + \sum_{m=1}^{\infty} P_{mn}^{jk} r_k^m \sin m\theta_k, \quad (3.5)$$

where

$$\left\{ \begin{array}{l} P_{mn}^{jk} \\ Q_{mn}^{jk} \end{array} \right\} = \frac{(-1)^{m+n}}{m!(n-1)!} \int_0^\infty \frac{\mu + K}{\mu - K} \mu^{m+n-1} e^{-2\mu f} \left\{ \begin{array}{l} \cos \mu(j-k)d \\ \sin \mu(j-k)d \end{array} \right\} d\mu \quad (3.6)$$

in an obvious notation. For computational purposes we note that

$$\begin{aligned} a^{m+n} P_{mn}^{jk} = \frac{(-1)^{m+n}}{m!(n-1)!} & \left[\int_0^{2Ka} \left((u + Ka) u^{m+n-1} e^{-2uf/a} \cos u(j-k)d/a \right. \right. \\ & \left. \left. - 2(Ka)^{m+n} e^{-2Kf} \cos Kd(j-k) \right) \frac{du}{u - Ka} \right. \\ & \left. + \int_{2Ka}^\infty \frac{u + Ka}{u - Ka} u^{m+n-1} e^{-2uf/a} \cos u(j-k)d/a du \right. \\ & \left. + 2\pi i (Ka)^{m+n} e^{-2Kf} \cos Kd(j-k) \right], \quad (3.7) \end{aligned}$$

where we have used the fact that

$$\int_0^{2Ka} \frac{du}{u - Ka} = 0. \quad (3.8)$$

Similarly for Q_{mn}^{jk} ; just replace \cos by \sin .

To re-expand the singular terms about $x = kd$, $z = -f$ we use the technique described in Linton & McIver (2001), §3.2.1 (see also the Appendix to O'Leary (1985)). We can thus show that for $j \neq k$, $r_k < \min(2f, |j-k|d)$,

$$\phi_n^j = \sum_{m=0}^{\infty} (P_{mn}^{jk} + C_{mn}^{jk}) r_k^m \cos m\theta_k - \sum_{m=1}^{\infty} (Q_{mn}^{jk} + S_{mn}^{jk}) r_k^m \sin m\theta_k, \quad (3.9)$$

$$\psi_n^j = \sum_{m=0}^{\infty} (Q_{mn}^{jk} - S_{mn}^{jk}) r_k^m \cos m\theta_k + \sum_{m=1}^{\infty} (P_{mn}^{jk} - C_{mn}^{jk}) r_k^m \sin m\theta_k, \quad (3.10)$$

where

$$\left\{ \begin{array}{l} C_{mn}^{jk} \\ S_{mn}^{jk} \end{array} \right\} = \frac{(n+m-1)!}{((j-k)d)^{n+m} m!(n-1)!} \left\{ \begin{array}{l} \cos[(n-m)\frac{\pi}{2}] \\ \sin[(n-m)\frac{\pi}{2}] \end{array} \right\}. \quad (3.11)$$

3.1. Infinite array

The case of an infinite array of identical submerged circular cylinders has been treated previously by Schnute (1967)[†] using an integral equation formulation as in Levine (1965). The method presented below is much simpler than that given in Schnute's thesis. We seek a solution in the form

$$\phi = \sum_{j=-\infty}^{\infty} e^{i\beta_j d} \sum_{n=1}^{\infty} a^n (\xi_n \phi_n^j + \eta_n \psi_n^j), \quad (3.12)$$

where, without loss of generality we can assume that $0 \leq \beta d < 2\pi$. It is straightforward to show that $\phi(x+d, z) = \exp(i\beta d)\phi(x, z)$ and so ϕ is quasi-periodic. Because of the symmetry of the geometry about $x=0$ we can also show that if there is a non-trivial solution for a particular value of βd , there will also be a solution with βd replaced by $2\pi - \beta d$ corresponding to a wave travelling in the opposite direction. Hence we can restrict attention to $0 \leq \beta d \leq \pi$.

On the assumption that there is a solution of this form, then from (3.3), for $z > f$,

$$\begin{aligned} \phi = \sum_{n=1}^{\infty} \frac{a^n (-1)^n}{(n-1)!} \sum_{j=-\infty}^{\infty} e^{i\beta_j d} \int_0^{\infty} \frac{\mu^{n-1} e^{-\mu f}}{\mu - K} (\mu \cosh \mu z + K \sinh \mu z) \\ \times \left((\xi_n + i\eta_n) e^{i\mu(x-jd)} + (\xi_n - i\eta_n) e^{-i\mu(x-jd)} \right) d\mu, \end{aligned}$$

which after application of the Poisson summation formula, writing $\beta_j = \beta + 2j\pi/d$, yields

$$\phi = \sum_{j=-\infty}^{\infty} \mathcal{A}_j \frac{(|\beta_j| \cosh \beta_j z + K \sinh |\beta_j| z)}{|\beta_j| - K} e^{i\beta_j x - |\beta_j| f}, \quad (3.13)$$

where

$$\mathcal{A}_j = \sum_{n=1}^{\infty} \frac{2\pi a^n (-1)^n}{d(n-1)!} |\beta_j|^{n-1} (\xi_n + \operatorname{sgn}(\beta_j) i\eta_n). \quad (3.14)$$

This requires that we assume $|\beta_j| \neq K$ for any j . However, we will see later that there is no singularity when this occurs since \mathcal{A}_j then vanishes. The solution can thus be represented as an infinite sum of modes, each having a different wavenumber and a different amplitude. The free surface elevation, which is given by $(i\omega/g)\phi|_{z=0}$, can be obtained from (3.13).

The form of the potential given by (3.12) satisfies all the conditions of the problem except the boundary condition on the cylinders. In fact, we only need to satisfy the boundary condition on one cylinder (which we take to be the one at the origin) and then all the others are satisfied because of quasi-periodicity. If we write

$$\begin{Bmatrix} \hat{P}_{mn} \\ \hat{Q}_{mn} \end{Bmatrix} = a^{n+m} \sum_{j=-\infty}^{\infty} e^{i\beta_j d} \begin{Bmatrix} P_{mn}^{j0} \\ Q_{mn}^{j0} \end{Bmatrix}, \quad \begin{Bmatrix} \hat{C}_{mn} \\ \hat{S}_{mn} \end{Bmatrix} = a^{n+m} \sum'_{j=-\infty}^{\infty} e^{i\beta_j d} \begin{Bmatrix} C_{mn}^{j0} \\ S_{mn}^{j0} \end{Bmatrix} \quad (3.15)$$

(the $j=0$ term is included in the definition of \hat{Q}_{mn} even though it is zero, and the dash on the summation indicates that the $j=0$ term should be omitted) then setting

[†] Thanks are due to Paul Martin for bringing this work to the attention of the author.

$\partial\phi/\partial r = 0$ on $r = a$ yields

$$\xi_m - \sum_{n=1}^{\infty} \left(\xi_n (\hat{P}_{mn} + \hat{C}_{mn}) + \eta_n (\hat{Q}_{mn} - \hat{S}_{mn}) \right) = 0, \quad m \geq 1, \quad (3.16)$$

$$\eta_m + \sum_{n=1}^{\infty} \left(\xi_n (\hat{Q}_{mn} + \hat{S}_{mn}) - \eta_n (\hat{P}_{mn} - \hat{C}_{mn}) \right) = 0, \quad m \geq 1. \quad (3.17)$$

The quantities \hat{P}_{mn} , \hat{Q}_{mn} , \hat{C}_{mn} and \hat{S}_{mn} can all be evaluated simply. In the case of \hat{C}_{mn} and \hat{S}_{mn} the series in (3.15) can actually be summed explicitly. If $n + m$ is odd, then $\hat{C}_{mn} = 0$. On the other hand, if $n + m$ is even (≥ 2) we have

$$\hat{C}_{mn} = \frac{(-1)^{n+1} (2\pi a/d)^{n+m}}{m!(n-1)!(n+m)} B_{n+m}(\beta d/2\pi), \quad (3.18)$$

where $B_n(\cdot)$ is a Bernoulli polynomial and we have used Abramowitz & Stegun (1965), eqn 23.1.18. This is valid for all βd in the interval $[0, 2\pi]$ and hence for all βd under consideration here. Similarly if $n + m$ is even then $\hat{S}_{mn} = 0$, whereas if $n + m$ is odd (≥ 1) we have

$$\hat{S}_{mn} = \frac{i(-1)^n (2\pi a/d)^{n+m}}{m!(n-1)!(n+m)} B_{n+m}(\beta d/2\pi) \quad (3.19)$$

(Abramowitz & Stegun (1965), eqn 23.1.17). Equation (3.19) is not valid when $m+n = 1$ and $\beta d = 0$ but this case is covered by the fact that $\hat{S}_{mn} = 0$ when $\beta d = 0$ or π and (3.19) is valid for all other βd in the interval $(0, 2\pi)$. The fact that \hat{C}_{mn} and \hat{S}_{mn} can be expressed so simply is a particularly attractive feature of the solution procedure.

Next \hat{P}_{mn} and \hat{Q}_{mn} can be expressed as exponentially convergent series using the Poisson summation formula. We obtain

$$\hat{P}_{mn} = \frac{(-a/d)^{m+n} \pi}{m!(n-1)!} \sum_{j=-\infty}^{\infty} \frac{|\beta_j| + K}{|\beta_j| - K} |\beta_j d|^{m+n-1} e^{-2|\beta_j|f}, \quad (3.20)$$

$$\hat{Q}_{mn} = \frac{(-a/d)^{m+n} \pi i}{m!(n-1)!} \sum_{j=-\infty}^{\infty} \operatorname{sgn}(\beta_j) \frac{|\beta_j| + K}{|\beta_j| - K} |\beta_j d|^{m+n-1} e^{-2|\beta_j|f}. \quad (3.21)$$

It is worth noting that the contribution from the pole at $\mu = K$ plays no role and this is consistent with the fact that the radiation condition which stipulates that the contour must pass beneath the pole is no longer relevant when there is an infinite array extending to $x = \pm\infty$. In deriving (3.20) and (3.21) we have to assume that $|\beta_j| \neq K$ for any j . However, as $|\beta_j| \rightarrow K$ for a particular j then \hat{P}_{mn} and \hat{Q}_{mn} clearly dominate in (3.16) and (3.17) and if we substitute the leading order behaviour from (3.20) and (3.21) it is straightforward to show that (3.16) and (3.17) each reduce to

$$\xi_n + i \operatorname{sgn}(\beta_j) \eta_n = 0, \quad n \geq 1, \quad (3.22)$$

which shows that \mathcal{A}_j , given by (3.14), is then zero.

To formulate (3.16) and (3.17) as a finite linear system we write

$$(x_1, x_2, \dots, x_{2N}) = (\xi_1, i\eta_1, \xi_2, i\eta_2, \dots, \xi_N, i\eta_N). \quad (3.23)$$

In other words $\xi_m = x_{2m-1}$ and $\eta_m = -ix_{2m}$. Then (3.16) and (3.17) become

$$x_{2m-1} - \sum_{n=1}^N \left(x_{2n-1} (\hat{P}_{mn} + \hat{C}_{mn}) - ix_{2n} (\hat{Q}_{mn} - \hat{S}_{mn}) \right) = 0, \quad 1 \leq m \leq N, \quad (3.24)$$

$$x_{2m} + \sum_{n=1}^N \left(ix_{2n-1} (\hat{Q}_{mn} + \hat{S}_{mn}) - x_{2n} (\hat{P}_{mn} - \hat{C}_{mn}) \right) = 0, \quad 1 \leq m \leq N, \quad (3.25)$$

and so we require the determinant of the $2N \times 2N$ matrix $\mathbf{M} = (M_{ij})_{i,j=1}^{2N}$ to vanish, where

$$\begin{aligned} M_{2m-1,2n-1} &= \delta_{mn} - \hat{P}_{mn} - \hat{C}_{mn}, & M_{2m-1,2n} &= i\hat{Q}_{mn} - i\hat{S}_{mn}, \\ M_{2m,2n-1} &= i\hat{Q}_{mn} + i\hat{S}_{mn}, & M_{2m,2n} &= \delta_{mn} - \hat{P}_{mn} + \hat{C}_{mn}, \end{aligned} \quad (3.26)$$

in which all the entries are real.

3.2. Finite array

We now seek a solution in the form

$$\phi = \phi_{\text{inc}} + \phi_{\text{sc}} = \phi_{\text{inc}} + \sum_{j=0}^{M-1} \sum_{n=1}^{\infty} a^n (\xi_n^j \phi_n^j + \eta_n^j \psi_n^j), \quad (3.27)$$

in which

$$\phi_{\text{inc}} = e^{iKx} e^{Kz} = e^{ikKd} e^{-Kf} \sum_{n=0}^{\infty} \frac{(-Kr_k)^n}{n!} e^{-in\theta_k}. \quad (3.28)$$

Re-expanding about $r_k = 0$ and then setting $\partial\phi/\partial r_k = 0$ on $r_k = a$ gives rise to

$$\xi_m^k - \sum_{n=1}^{\infty} \sum_{j=0}^{M-1} a^{m+n} (\xi_n^j (P_{mn}^{jk} + C_{mn}^{jk}) + \eta_n^j (Q_{mn}^{jk} - S_{mn}^{jk})) = e^{ikKd} e^{-Kf} \frac{(-Ka)^m}{m!}, \quad (3.29)$$

$$\eta_m^k + \sum_{n=1}^{\infty} \sum_{j=0}^{M-1} a^{m+n} (\xi_n^j (Q_{mn}^{jk} + S_{mn}^{jk}) - \eta_n^j (P_{mn}^{jk} - C_{mn}^{jk})) = -ie^{ikKd} e^{-Kf} \frac{(-Ka)^m}{m!}, \quad (3.30)$$

with $m \geq 1, k = 0, \dots, M-1$ in each case. For convenience, we have defined $C_{mn}^{kk} = S_{mn}^{kk} = 0$. This can be truncated easily by restricting m and n to range between 1 and N , which yields a $2MN \times 2MN$ system of equations.

From (3.27) and Linton & McIver (2001), eqns B.18 & B.27, we have that as $x \rightarrow \pm\infty$

$$\phi \sim e^{iKx} e^{Kz} + 2\pi e^{\pm iKx} e^{K(z-f)} \sum_{j=0}^{M-1} \sum_{n=1}^{\infty} \frac{(-Ka)^n}{(n-1)!} e^{\mp iKjd} (i\xi_n^j \mp \eta_n^j). \quad (3.31)$$

Hence the reflection and transmission coefficients are

$$R_M = 2\pi e^{-Kf} \sum_{j=0}^{M-1} \sum_{n=1}^{\infty} \frac{(-Ka)^n}{(n-1)!} e^{iKjd} (i\xi_n^j + \eta_n^j), \quad (3.32)$$

$$T_M = 1 + 2\pi e^{-Kf} \sum_{j=0}^{M-1} \sum_{n=1}^{\infty} \frac{(-Ka)^n}{(n-1)!} e^{-iKjd} (i\xi_n^j - \eta_n^j). \quad (3.33)$$

Alternative expressions for these coefficients can be obtained by determining the horizontal and vertical exciting forces on the cylinders and then using standard reciprocity relations (see, for example, Linton & McIver (2001), §1.4). The array is not symmetric about $x = 0$, but if this is accounted for we can show that

$$R_M = \frac{1}{2} (Z/\bar{Z} + X/\bar{X}), \quad T_M = \frac{1}{2} e^{-i(M-1)Kd} (Z/\bar{Z} - X/\bar{X}), \quad (3.34)$$

where we have written

$$X = \sum_{j=0}^{M-1} \eta_1^j, \quad Z = \sum_{j=0}^{M-1} \xi_1^j, \quad (3.35)$$

quantities which are proportional to, respectively, the horizontal and vertical exciting forces on the cylinder array.

For a single cylinder, it follows from (3.32) that $R_1 = 0$ since in that case (3.29) and (3.30) yield $\eta_m^0 = -i\xi_m^0$ for all m . The fact that in general $R_M \neq 0$ for $M > 1$ is hardly surprising, though this seems to have been established first in Schnute (1971), where the case $M = 2$ was considered. An analysis based on matched asymptotic expansions was performed in McIver (1990) under the assumption that the wavelength is much greater than the cylinder spacing which in turn is much greater than the cylinder radius, leading to the conclusion that $R_M = O((Kd)^2(a/d)^4)$ in this limit.

4. Full linear theory: oblique waves

Multipole expansions have been used previously to solve scattering problems involving oblique wave interactions with finite arrays of submerged, horizontal circular cylinders in Shen & Zheng (2007). In fact that article considered the case of finite water depth. Nevertheless much of the detail remains the same and only brief details will be here, focussing on the infinite array case.

In the case of a finite array we will take as our incident field a plane wave making an angle $\varphi \in (0, \pi/2)$ with the positive x -axis given by

$$\phi_{\text{inc}} = e^{i\alpha x} e^{Kz}, \quad (4.1)$$

(having suppressed the $\exp(i\ell y)$ dependence) where

$$\alpha = K \cos \varphi, \quad \ell = K \sin \varphi. \quad (4.2)$$

In particular, we note that in this case $0 < \ell < K$. However, for an infinite array there is no such upper limit for ℓ . For a finite array we could also consider the case of $\ell > K$ with no incident wave and look for trapped modes, as in Porter & Evans (1998).

Multipoles singular at $r_j = 0$ and symmetric in θ_j are now (Linton & McIver (2001), eqn B.111)

$$\phi_n^j = K_n(\ell r_j) \cos n\theta_j + (-1)^n \int_0^\infty \frac{\nu + K}{\nu - K} \cosh n\mu \cos(\lambda(x - jd)) e^{\nu(z-f)} d\mu, \quad n \geq 0, \quad (4.3)$$

where $\nu = \ell \cosh \mu$, $\lambda = \ell \sinh \mu$, and multipoles singular at $r_j = 0$ and antisymmetric in θ_j are (Linton & McIver (2001), eqn B.117)

$$\psi_n^j = K_n(\ell r_j) \sin n\theta_j - (-1)^n \int_0^\infty \frac{\nu + K}{\nu - K} \sinh n\mu \sin(\lambda(x - jd)) e^{\nu(z-f)} d\mu, \quad n \geq 1. \quad (4.4)$$

Here $K_n(\cdot)$ (and $I_n(\cdot)$ which appears below) are modified Bessel functions. For $z > f$ we also have (Linton & McIver (2001), eqn B.101)

$$\phi_n^j = 2(-1)^n \int_0^\infty \frac{e^{-\nu f}}{\nu - K} (\nu \cosh \nu z + K \sinh \nu z) \cosh n\mu \cos(\lambda(x - jd)) d\mu, \quad (4.5)$$

with ψ_n^j given by the same expression after replacing $\cosh n\mu$ by $\sinh n\mu$ and \cos by $-\sin$.

As before we can develop series expansions for these multipoles about the point $x = kd$, $z = -f$. For all j, k we have, for $r < 2f$,

$$\phi_n^j = K_n(\ell r_j) \cos n\theta_j + \sum_{m=0}^{\infty} P_{mn}^{jk} I_m(\ell r_k) \cos m\theta_k - \sum_{m=1}^{\infty} Q_{mn}^{jk} I_m(\ell r_k) \sin m\theta_k, \quad (4.6)$$

$$\psi_n^j = K_n(\ell r_j) \sin n\theta_j + \sum_{m=0}^{\infty} R_{mn}^{jk} I_m(\ell r_k) \cos m\theta_k + \sum_{m=1}^{\infty} S_{mn}^{jk} I_m(\ell r_k) \sin m\theta_k, \quad (4.7)$$

where

$$P_{mn}^{jk} = \epsilon_m (-1)^{m+n} \int_0^\infty \frac{\nu + K}{\nu - K} \cosh m\mu \cosh n\mu \cos[\lambda d(j - k)] e^{-2\nu f} d\mu, \quad (4.8)$$

$$Q_{mn}^{jk} = \frac{2}{\epsilon_n} R_{nm}^{jk} = 2(-1)^{m+n} \int_0^\infty \frac{\nu + K}{\nu - K} \sinh m\mu \cosh n\mu \sin[\lambda d(j - k)] e^{-2\nu f} d\mu, \quad (4.9)$$

$$S_{mn}^{jk} = 2(-1)^{m+n} \int_0^\infty \frac{\nu + K}{\nu - K} \sinh m\mu \sinh n\mu \cos[\lambda d(j - k)] e^{-2\nu f} d\mu. \quad (4.10)$$

Here $\epsilon_0 = 1$, $\epsilon_n = 2$ for $n \geq 1$. For computational purposes we note that

$$\begin{aligned} P_{mn}^{jk} = \epsilon_m (-1)^{m+n} \left[\int_0^b \left((\nu + K) \lambda^{-1} \cosh m\mu \cosh n\mu \cos[\lambda d(j - k)] e^{-2\nu f} \right. \right. \\ \left. \left. - 2(K/\alpha) \cosh m\gamma \cosh n\gamma \cos[\alpha d(j - k)] e^{-2Kf} \right) \frac{\lambda d\mu}{\nu - K} \right. \\ \left. + \int_b^\infty \frac{\nu + K}{\nu - K} \cosh m\mu \cosh n\mu \cos[\lambda d(j - k)] e^{-2\nu f} d\mu \right. \\ \left. + 2\pi i (K/\alpha) \cosh m\gamma \cosh n\gamma \cos[\alpha d(j - k)] e^{-2Kf} \right], \quad (4.11) \end{aligned}$$

where $K = \ell \cosh \gamma$ and $\cosh b = (2K/\ell) - 1$. Similarly for Q_{mn}^{jk} , R_{mn}^{jk} and S_{mn}^{jk} .

To re-expand the singular terms about $x = kd$, $z = -f$ we use Graf's addition theorem which here takes the form

$$K_n(\ell r_j) e^{in\theta_j} = \sum_{m=-\infty}^{\infty} i^{m+n} K_{n-m}(|j - k|\ell d) e^{i(n-m)\chi_{jk}} I_m(\ell r_k) e^{im\theta_k}, \quad j \neq k. \quad (4.12)$$

We can thus show that, for $j \neq k$ and $r_k < \min(2f, |j - k|d)$,

$$\phi_n^j = \sum_{m=0}^{\infty} (P_{mn}^{jk} + D_{mn}^{jk}) I_m(\ell r_k) \cos m\theta_k + \sum_{m=1}^{\infty} (E_{mn}^{jk} - Q_{mn}^{jk}) I_m(\ell r_k) \sin m\theta_k, \quad (4.13)$$

$$\psi_n^j = \sum_{m=0}^{\infty} (R_{mn}^{jk} + F_{mn}^{jk}) I_m(\ell r_k) \cos m\theta_k + \sum_{m=1}^{\infty} (S_{mn}^{jk} + G_{mn}^{jk}) I_m(\ell r_k) \sin m\theta_k, \quad (4.14)$$

where, with $K_n^{jk} = K_n(|j - k|\ell d)$,

$$D_{mn}^{jk} = \frac{\epsilon_m}{2} \left(\cos[(n+m)\frac{\pi}{2}]K_{n-m}^{jk} + \cos[(n-m)\frac{\pi}{2}]K_{n+m}^{jk} \right), \quad (4.15)$$

$$E_{mn}^{jk} = \text{sgn}(k-j) \left(-\sin[(n+m)\frac{\pi}{2}]K_{n-m}^{jk} + \sin[(n-m)\frac{\pi}{2}]K_{n+m}^{jk} \right), \quad (4.16)$$

$$F_{mn}^{jk} = \frac{\epsilon_m}{2} \text{sgn}(k-j) \left(\sin[(n+m)\frac{\pi}{2}]K_{n-m}^{jk} + \sin[(n-m)\frac{\pi}{2}]K_{n+m}^{jk} \right), \quad (4.17)$$

$$G_{mn}^{jk} = \cos[(n+m)\frac{\pi}{2}]K_{n-m}^{jk} - \cos[(n-m)\frac{\pi}{2}]K_{n+m}^{jk}. \quad (4.18)$$

Note that $D_{mn}^{jk} = G_{mn}^{jk} = 0$ if $m+n$ is odd while $E_{mn}^{jk} = F_{mn}^{jk} = 0$ if $m+n$ is even.

4.1. Infinite array

We seek a solution in the form

$$\phi = \sum_{j=-\infty}^{\infty} e^{i\beta_j d} \left(\sum_{n=0}^{\infty} \tilde{\xi}_n \phi_n^j + \sum_{n=1}^{\infty} \tilde{\eta}_n \psi_n^j \right), \quad (4.19)$$

with $0 \leq \beta d \leq \pi$ as before. Incorporating (4.5) and using the Poisson summation formula shows that, for $z > f$, provided $\nu_j \neq K$ for any j ,

$$\phi = \sum_{j=-\infty}^{\infty} \mathcal{A}_j \frac{\nu_j \cosh \nu_j z + K \sinh \nu_j z}{\nu_j(\nu_j - K)} e^{i\beta_j x - \nu_j f}, \quad (4.20)$$

where

$$\mathcal{A}_j = \sum_{n=0}^{\infty} \frac{2\pi}{d} (-1)^n (\tilde{\xi}_n \cosh n\mu_j + i\tilde{\eta}_n \sinh n\mu_j). \quad (4.21)$$

Here

$$\mu_j = \sinh^{-1}(\beta_j/\ell) = \ln \left[\frac{\beta_j}{\ell} + \left(\frac{\beta_j^2}{\ell^2} + 1 \right)^{1/2} \right] \quad (4.22)$$

and we have written

$$\nu_j = \ell \cosh \mu_j = (\ell^2 + \beta_j^2)^{1/2}. \quad (4.23)$$

The apparent singularities at $\nu_j = K$ are removable, since it can be shown that if $\nu_j = K$ for a particular j , then $\mathcal{A}_j = 0$.

As in the $\ell = 0$ case, we only need to satisfy the boundary condition on one cylinder (which we take to be the one at the origin) and then all the others are satisfied because of quasi-periodicity. We define

$$\hat{P}_{mn} = \sum_{j=-\infty}^{\infty} e^{i\beta_j d} P_{mn}^{j0} \quad (4.24)$$

and similarly for \hat{Q}_{mn} , \hat{R}_{mn} and \hat{S}_{mn} , and

$$\hat{D}_{mn} = \sum_{j=-\infty}^{\infty} e^{i\beta_j d} D_{mn}^{j0} \quad (4.25)$$

and similarly for \hat{E}_{mn} , \hat{F}_{mn} and \hat{G}_{mn} . Setting $\partial\phi/\partial r = 0$ on $r = a$ then yields

$$\xi_m + \sum_{n=0}^{\infty} Z_{mn} \xi_n (\hat{P}_{mn} + \hat{D}_{mn}) + \sum_{n=1}^{\infty} Z_{mn} \eta_n (\hat{R}_{mn} + \hat{F}_{mn}) = 0, \quad m \geq 0, \quad (4.26)$$

$$\eta_m + \sum_{n=0}^{\infty} Z_{mn} \xi_n (\hat{E}_{mn} - \hat{Q}_{mn}) + \sum_{n=1}^{\infty} Z_{mn} \eta_n (\hat{S}_{mn} + \hat{G}_{mn}) = 0, \quad m \geq 1, \quad (4.27)$$

where we have written $\xi_m = \tilde{\xi}_m K'_m(\ell a)$, $\eta_m = \tilde{\eta}_m K'_m(\ell a)$ and $Z_{mn} = I'_m(\ell a)/K'_n(\ell a)$.

It is helpful to define

$$\hat{K}_n = \begin{cases} 2 \sum_{j=1}^{\infty} K_n(j\ell d) \cos \beta j d, & n \text{ even,} \\ -2i \sum_{j=1}^{\infty} K_n(j\ell d) \sin \beta j d, & n \text{ odd.} \end{cases} \quad (4.28)$$

If $n + m$ is odd, then $\hat{D}_{mn} = \hat{G}_{mn} = 0$. On the other hand, if $n + m$ is even we have

$$\hat{D}_{mn} = \frac{\epsilon_m}{2} \left[(-1)^{(n+m)/2} \hat{K}_{n-m} + (-1)^{(n-m)/2} \hat{K}_{n+m} \right], \quad (4.29)$$

$$\hat{G}_{mn} = (-1)^{(n+m)/2} \hat{K}_{n-m} - (-1)^{(n-m)/2} \hat{K}_{n+m} \quad (4.30)$$

and these are clearly real. Similarly if $n + m$ is even, then $\hat{E}_{mn} = \hat{F}_{mn} = 0$, whereas if $n + m$ is odd we have

$$\hat{E}_{mn} = -(-1)^{(n+m-1)/2} \hat{K}_{n-m} + (-1)^{(n-m-1)/2} \hat{K}_{n+m}, \quad (4.31)$$

$$\hat{F}_{mn} = \frac{\epsilon_m}{2} \left[(-1)^{(n+m-1)/2} \hat{K}_{n-m} + (-1)^{(n-m-1)/2} \hat{K}_{n+m} \right], \quad (4.32)$$

which are imaginary. Application of the Poisson summation formula shows that

$$\hat{P}_{mn} = \epsilon_m \pi (-1)^{m+n} \sum_{j=-\infty}^{\infty} \frac{\nu_j + K}{\nu_j - K} \cosh m \mu_j \cosh n \mu_j \frac{e^{-2\nu_j f}}{\nu_j d}, \quad (4.33)$$

$$\hat{Q}_{mn} = \frac{2}{\epsilon_n} \hat{R}_{nm} = 2\pi i (-1)^{m+n} \sum_{j=-\infty}^{\infty} \frac{\nu_j + K}{\nu_j - K} \sinh m \mu_j \cosh n \mu_j \frac{e^{-2\nu_j f}}{\nu_j d}, \quad (4.34)$$

$$\hat{S}_{mn} = 2\pi (-1)^{m+n} \sum_{j=-\infty}^{\infty} \frac{\nu_j + K}{\nu_j - K} \sinh m \mu_j \sinh n \mu_j \frac{e^{-2\nu_j f}}{\nu_j d}. \quad (4.35)$$

All the quantities in (4.26) and (4.27) are then given in terms of exponentially convergent series.

To formulate (4.26) and (4.27) as a finite linear system we write

$$(x_1, x_2, \dots, x_{2N}) = (\xi_0, i\eta_1, \xi_1, i\eta_2, \dots, \xi_{N-1}, i\eta_N). \quad (4.36)$$

In other words $\xi_m = x_{2m+1}$ and $\eta_m = -ix_{2m}$. Then (4.26) and (4.27) become

$$x_{2m+1} + \sum_{n=0}^{N-1} Z_{mn} x_{2n+1} (\hat{P}_{mn} + \hat{D}_{mn}) - i \sum_{n=1}^N Z_{mn} x_{2n} (\hat{R}_{mn} + \hat{F}_{mn}) = 0, \quad (4.37)$$

$$0 \leq m \leq N-1,$$

$$x_{2m} + i \sum_{n=0}^{N-1} Z_{mn} x_{2n+1} (\hat{E}_{mn} - \hat{Q}_{mn}) + \sum_{n=1}^N Z_{mn} x_{2n} (\hat{S}_{mn} + \hat{G}_{mn}) = 0, \quad (4.38)$$

$$1 \leq m \leq N.$$

Thus we require the determinant of the $2N \times 2N$ matrix $\mathbf{M} = (M_{ij})_{i,j=1}^{2N}$ to vanish, where

$$\begin{aligned} M_{2m-1,2n-1} &= \delta_{mn} + Z_{m-1,n-1}(\hat{P}_{m-1,n-1} + \hat{D}_{m-1,n-1}), \\ M_{2m-1,2n} &= -iZ_{m-1,n}(\hat{R}_{m-1,n} + \hat{F}_{m-1,n}), \\ M_{2m,2n-1} &= iZ_{m,n-1}(\hat{E}_{m,n-1} - \hat{Q}_{m,n-1}), \\ M_{2m,2n} &= \delta_{mn} + Z_{mn}(\hat{S}_{mn} + \hat{G}_{mn}), \end{aligned} \quad (4.39)$$

in which all the entries are real. It can be shown that as $\ell \rightarrow 0$ the system for oblique waves reduces to that for $\ell = 0$.

4.2. Finite array

We seek a solution in the form

$$\phi = \phi_{\text{inc}} + \phi_{\text{sc}} = \phi_{\text{inc}} + \sum_{j=0}^{M-1} \left(\sum_{n=0}^{\infty} \tilde{\xi}_n^j \phi_n^j + \sum_{n=1}^{\infty} \tilde{\eta}_n^j \psi_n^j \right), \quad (4.40)$$

in which (see Linton & McIver (2001), eqn 3.21)

$$\phi_{\text{inc}} = e^{i\alpha x} e^{Kz} = e^{ik\alpha d} e^{-Kf} \sum_{n=0}^{\infty} \epsilon_n (-1)^n \text{I}_n(\ell r_k) \cos n(\theta_k + i\gamma). \quad (4.41)$$

Setting $\partial\phi/\partial r_k = 0$ on $r = a_k$ then yields

$$\begin{aligned} \xi_m^k + \sum_{j=0}^{M-1} \left(\sum_{n=0}^{\infty} Z_{mn} \xi_n^j (P_{mn}^{jk} + D_{mn}^{jk}) + \sum_{n=1}^{\infty} Z_{mn} \eta_n^j (R_{mn}^{jk} + F_{mn}^{jk}) \right) \\ = -\epsilon_m (-1)^m e^{ik\alpha d} e^{-Kf} \text{I}'_m(\ell a) \cosh m\gamma, \quad m \geq 0, k = 0, \dots, M-1, \end{aligned} \quad (4.42)$$

$$\begin{aligned} \eta_m^k + \sum_{j=0}^{M-1} \left(\sum_{n=0}^{\infty} Z_{mn} \xi_n^j (E_{mn}^{jk} - Q_{mn}^{jk}) + \sum_{n=1}^{\infty} Z_{mn} \eta_n^j (S_{mn}^{jk} + G_{mn}^{jk}) \right) \\ = 2i(-1)^m e^{ik\alpha d} e^{-Kf} \text{I}'_m(\ell a) \sinh m\gamma, \quad m \geq 1, k = 0, \dots, M-1, \end{aligned} \quad (4.43)$$

where we have defined $D_{mn}^{kk} = E_{mn}^{kk} = F_{mn}^{kk} = G_{mn}^{kk} = 0$. This can readily be truncated to a $2MN \times 2MN$ system by restricting m and n in (4.42) and (4.43) to range between 0 and $N-1$ or 1 and N as appropriate.

From (4.40) and Linton & McIver (2001), eqns B.112 & B.118, we have that as $x \rightarrow \pm\infty$

$$\phi_{\text{sc}} \sim \frac{2\pi K}{\alpha} e^{K(z-f)} \sum_{j=0}^{M-1} e^{\pm i\alpha(x-jd)} \left(\sum_{n=0}^{\infty} i(-1)^n \tilde{\xi}_n^j \cosh n\gamma \mp \sum_{n=1}^{\infty} (-1)^n \tilde{\eta}_n^j \sinh n\gamma \right). \quad (4.44)$$

Hence the reflection and transmission coefficients are

$$R_M = \frac{2\pi K}{\alpha} e^{-Kf} \sum_{j=0}^{M-1} e^{ij\alpha d} \left(\sum_{n=0}^{\infty} i(-1)^n \tilde{\xi}_n^j \cosh n\gamma + \sum_{n=1}^{\infty} (-1)^n \tilde{\eta}_n^j \sinh n\gamma \right), \quad (4.45)$$

$$T_M = 1 + \frac{2\pi K}{\alpha} e^{-Kf} \sum_{j=0}^{M-1} e^{-ij\alpha d} \left(\sum_{n=0}^{\infty} i(-1)^n \tilde{\xi}_n^j \cosh n\gamma - \sum_{n=1}^{\infty} (-1)^n \tilde{\eta}_n^j \sinh n\gamma \right). \quad (4.46)$$

As in §3, alternative expressions for the reflection and transmission coefficients can be obtained in terms of the horizontal and vertical exciting forces on the cylinders. In fact (3.34) remains true in this case if Kd is replaced by αd in the expression for T_M .

5. Numerical results

The size of the systems of equations derived in §§3 and 4 involve the truncation parameter N . In all the results presented below we have used $N = 8$. This means that 16 multipoles are used to represent the scattered field generated by each cylinder. For all the parameter values that we consider, which does not include the cases of high frequencies or cylinders very close to the free surface or to each other, this is more than sufficient to guarantee high accuracy. Results are presented for up to 60 cylinders, a case which thus requires the solution to a 960×960 system of equations. Reflection and transmission coefficients were always checked by calculating them both from the series expressions and in terms of the exciting forces, and energy conservation was always satisfied to a great degree of accuracy.

We begin with the $\ell = 0$ case. The transfer matrix analysis in §2 (based on a wide-spacing approximation) suggests that Bragg-type resonance is unlikely for submerged cylinders in deep water because, as is well known, $|R| = 0$ for a single submerged cylinder for any submergence depth and any frequency (Dean (1948)). That resonant reflection *does* occur is clear from Figure 1 which shows the modulus of the reflection coefficient as a function of Ka for 5, 10 and 20 cylinders when $f/d = 0.375$ and $a/d = 0.25$. A wavelength of twice the geometrical period would correspond to $Ka = \pi/4$ in this case whereas the figure shows that the first resonance actually occurs for $Ka \approx 0.57$ which corresponds to a wavelength of roughly 2.75 times the geometrical period. This shift is consistent with (2.17) if we interpret this condition as suggesting that resonance is most likely to occur when $|\cos \chi| = 1$. When $Ka = 0.57$ we can compute the phase of the transmission coefficient as approximately 0.93 which leads to a value of χ which is just greater than π .

To see whether it is possible to have $|T_M| \rightarrow 0$ as $M \rightarrow \infty$ we compute dispersion curves based on solving (3.24) and (3.25). We fix Kd and search for values of $\beta d \in [0, \pi]$ for which a non-trivial solution to this coupled system exists. Results for four pairs of values of f/d and a/d are shown in Figures 2–3 and in Table 1. Figure 2 shows dispersion curves over the range $0 < Kd < 8$, together with the zig-zag line made up from the curves $\beta d = Kd$, $\beta d = 2\pi - Kd$ ($\beta_{-1} = -K$) and $\beta d = Kd - 2\pi$ ($\beta_1 = K$), which all the other curves essentially follow. The first ‘point’ on this zig-zag line is at $Kd = \pi$ which corresponds to a wavelength of exactly twice the geometrical period. All the curves shown exhibit band gaps. In other words there are ranges of values of Kd for which there are no allowable values of βd and these intervals are shown in close-up in Figure 3 and collected together in Table 1. If we consider the case $f/d = 0.375$, $a/d = 0.25$ (which corresponds to the parameters in Figure 1) then we see that the resonances in Figure 1 occur at values of Kd (≈ 2.28 and 5.5) in the middle of the band gaps.

The existence of these band gaps means that we would expect $|T_M| \rightarrow 0$ as $M \rightarrow \infty$ whenever Kd lies in one of the gaps and this is indeed the case, as shown in Figure 4, which in fact shows the transmitted energy $|T_M|^2$ as a function of M for a range of values of Kd (and with $f/d = 0.375$ and $a/d = 0.25$ as in Figure 1). Three of these Kd values lie within the band gap (see Table 1) and for these the energy decays monotonically as M increases. For all the other values of Kd the transmitted energy oscillates between 1 and some finite value, that value (and the frequency of the oscillations) decreasing as the edge of the band gap is approached.

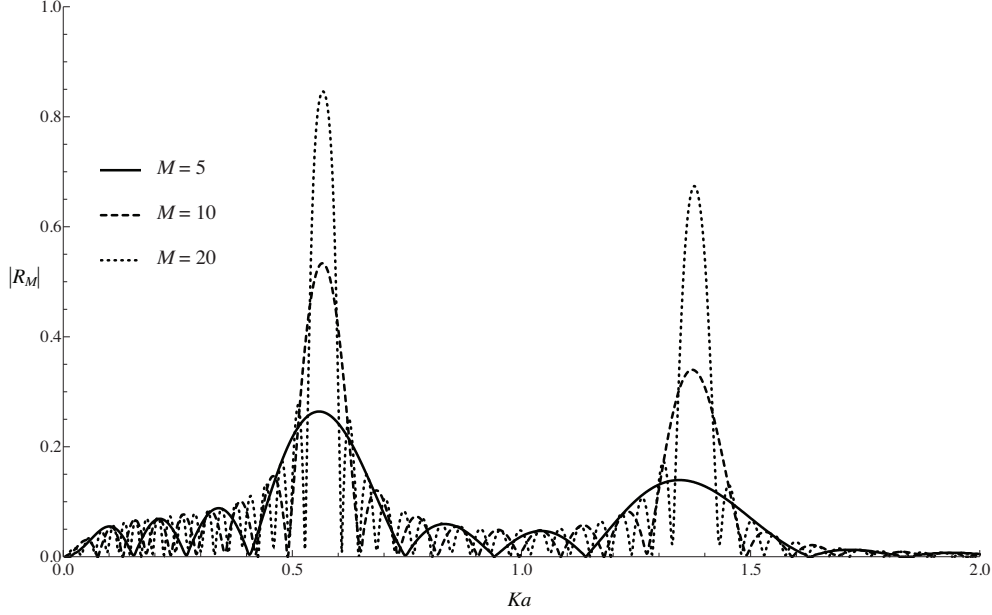


FIGURE 1. Modulus of reflection coefficient as a function of Ka for $f/d = 0.375$, $a/d = 0.25$.

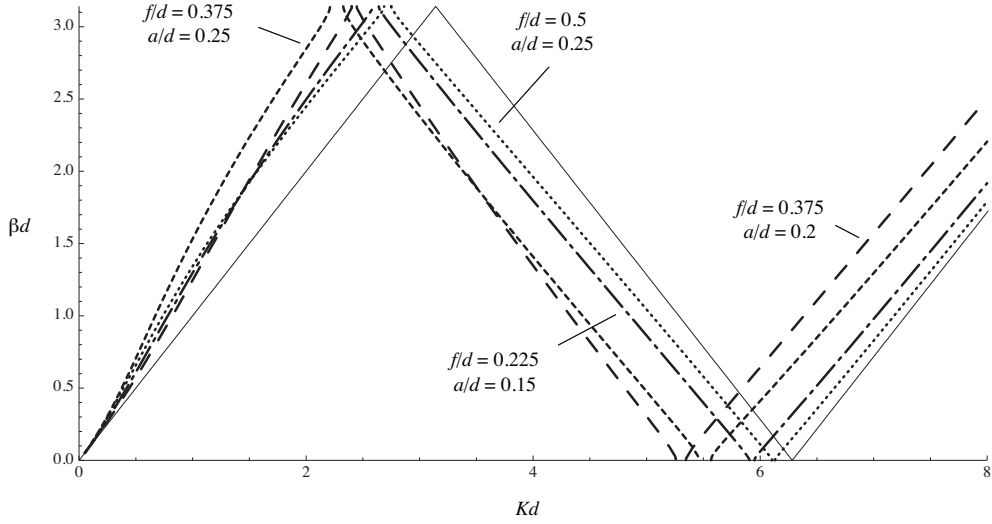


FIGURE 2. Dispersion curves showing βd as a function of Kd . The thin solid lines are $\beta d = Kd$, $\beta d = 2\pi - Kd$ and $\beta d = Kd - 2\pi$. Dotted lines are for $f/d = 0.5$, $a/d = 0.25$. Shorter dashed lines are for $f/d = 0.375$, $a/d = 0.25$. Longer dashed lines are for $f/d = 0.375$, $a/d = 0.2$. Dash-dot lines are for $f/d = 0.225$, $a/d = 0.15$.

Next we turn to the case $\ell > 0$. Figure 5 shows dispersion curves βd as a function of Kd for various values of ℓd based on solving (4.37) and (4.38). The most obvious difference between this and the case $\ell = 0$ (Figure 2) is that as ℓ increases from zero a gap appears at the origin. Thus, for a given $\ell > 0$ there is a minimum value of the frequency for which modes are possible. However, the minimum value of Kd is less than ℓd in all cases that we have computed and so it is not possible to consider a scattering problem corresponding to a frequency in this gap. Computed values for the band gaps shown in Figure 5 are

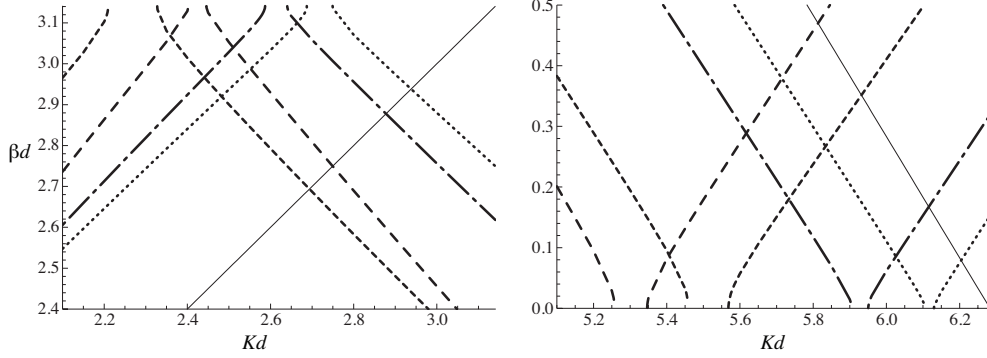
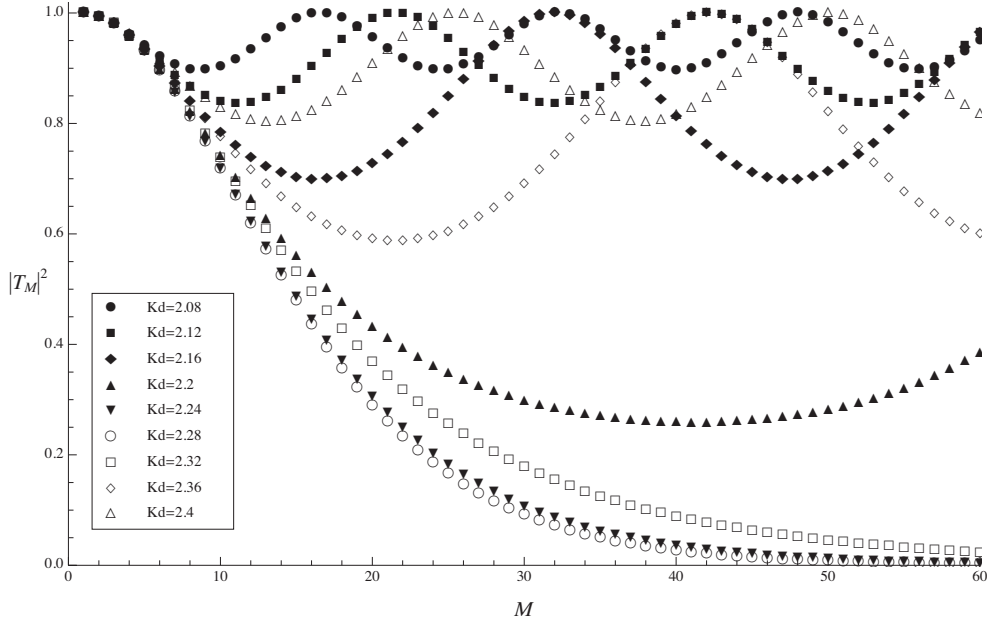


FIGURE 3. Close-up of the first two band gaps.

f/d	a/d	gap 1 (Kd)	gap 2 (Kd)
0.5	0.25	(2.687, 2.794)	(6.105, 6.130)
0.375	0.25	(2.210, 2.328)	(5.457, 5.569)
0.375	0.2	(2.587, 2.640)	(5.904, 5.951)
0.225	0.15	(2.402, 2.446)	(5.257, 5.347)

TABLE 1. Computed values for band-gaps shown in Figures 2 and 3.

FIGURE 4. Transmitted energy as a function of the number of cylinders in the array for $f/d = 0.375$, $a/d = 0.25$.

shown in Table 2. For the second and third band gaps we also show intervals of Kd in which the approximate condition for stop bands (2.17) is satisfied. This can only be done when $K > \ell$ which is why no approximate values appear for the first band gap. The

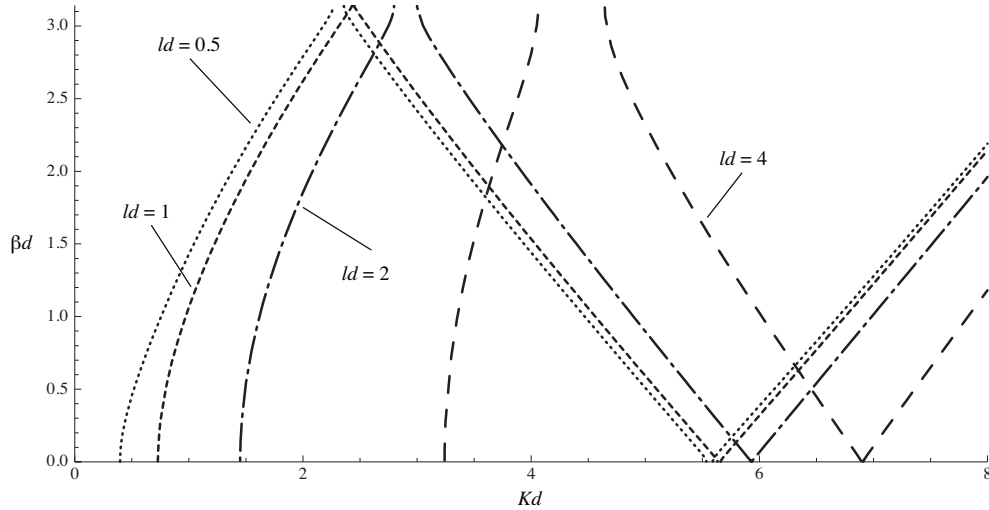


FIGURE 5. Dispersion curves showing βd as a function of Kd . Dotted lines are for $\ell d = 0.5$; shorter dashed lines are for $\ell d = 1$; dash-dot lines are for $\ell d = 2$; longer dashed lines are for $\ell d = 4$. In all cases $f/d = 0.375$, $a/d = 0.25$.

ℓd	gap 1 (Kd)	gap 2 (Kd)	gap 3 (Kd)
0.5	(0, 0.396)	(2.266, 2.358) (2.262, 2.270)	(5.535, 5.592) (5.592, 5.595)
1	(0, 0.728)	(2.429, 2.448) (2.377, 2.431)	(5.635, 5.661) (5.661, 5.671)
2	(0, 1.450)	(2.798, 3.000) (2.765, 3.001)	(5.930, 5.931) (5.931, 5.943)
4	(0, 3.240)	(4.061, 4.644) (4.057, 4.645)	(6.893, 6.911) (6.896, 6.910)

TABLE 2. Computed values for the band-gaps shown in Figure 5. Where there are two intervals listed, the lower values are computed from the approximate condition (2.17).

transfer matrix analysis clearly provides an excellent guide as to the location of the band gaps.

The relation between these band gaps and scattering problems for oblique waves in which $|T_M| \rightarrow 0$ as $M \rightarrow \infty$ is less clear than in the case $\ell = 0$. This is because in such a scattering problem one would usually treat either the incident wave angle or the frequency as fixed and then vary the other of these two parameters; in which case $\ell = K \sin \varphi$ is not constant. Nevertheless, the association can still be made. Figure 6 shows the transmitted energy $|T_M|^2$ as a function of M for a range of values of Kd when $f/d = 0.375$, $a/d = 0.25$ and the incident wave angle φ is fixed at 45° . Two of the values of Kd used in the Figure appear to lead to a decaying transmission coefficient, namely $Kd = 2.8$ and $Kd = 3$. Taking the latter as an example, we have $\ell d = 3/\sqrt{2} \approx 2.12$ and if we compute the band gap for $\ell d = 2.12$ we find that it is $(2.854, 3.084)$, which includes $Kd = 3$.

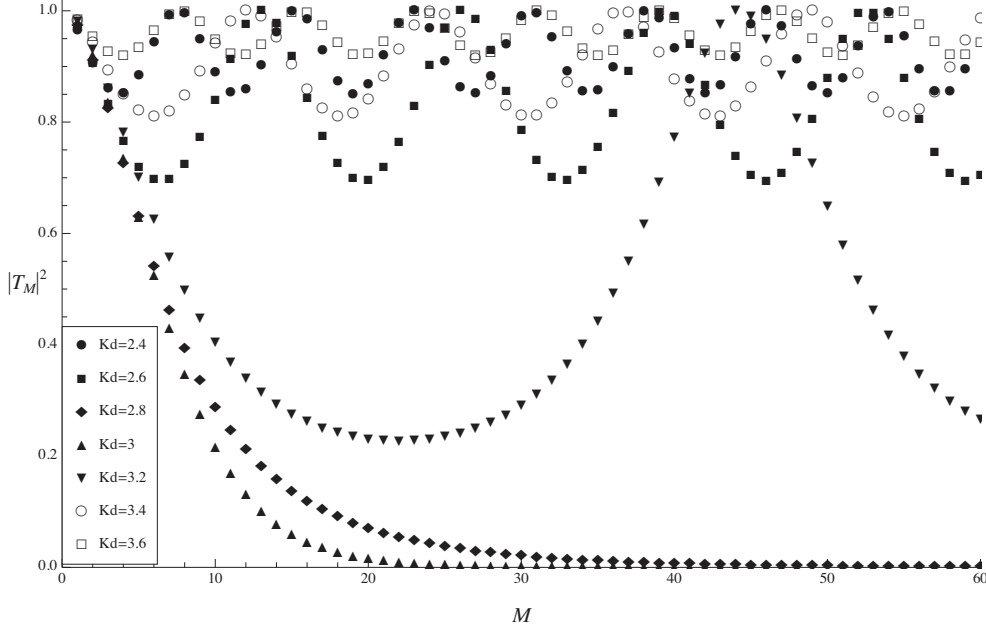


FIGURE 6. Transmitted energy as a function of the number of cylinders in the array for $f/d = 0.375$, $a/d = 0.25$ and $\varphi = 45^\circ$.

We can also plot dispersion curves showing βd as a function of ℓd for various values of Kd . From the general structure of the dispersion curves shown in Figure 5 we can expect that there will be a different structure to the βd - ℓd curves depending on the size of Kd , with more band gaps appearing as Kd increases. This is borne out by the curves shown in Figure 7, which is for the case $f/d = 0.5$, $a/d = 0.25$. When $Kd = 2$ there are no band gaps, for $Kd = 3, 4, 5$ and 6 there is one band gap, while for $Kd = 7$ there are two.

The curves in Figure 7 allow us to determine the direction in which the energy associated with these modes travels. If we write $\mathbf{k} = (\beta, \ell)$ for the wavenumber vector then the group velocity for any mode is simply $d\omega/d\mathbf{k}$ and since the curves in Figure 7 are constant frequency contours, the group velocity is always perpendicular to these curves, in the direction of increasing Kd . All the curves hit the $\ell d = 0$ axis at right angles, as they must since then there is no y -variation and the modes propagate along the x -axis. (The figure shows only half the picture; one needs to reflect about either $\beta d = 0$ or $\beta d = \pi$ to see all the possible modes.) When $Kd = 2$ the direction of the modes gradually shifts round until the mode cuts off, at which point propagation is entirely in the y -direction. For higher values of Kd we can have fundamentally different modes which have the same frequency and which propagate in the same direction. Thus, for example, when $Kd = 4$ there are three modes for which the energy propagates in the y -direction; two for which $\beta d = \pi$ (on either side of the band gap) and one for which $\beta d = 0$.

In Figure 7, the band gap for $Kd = 4$ corresponds to $\ell d \in (2.808, 3.017)$, which is equivalent to $\varphi \in (44.6^\circ, 49.0^\circ)$. Thus if we fix the frequency at $Kd = 4$ and vary the incident wave angle we would expect $|T_M| \rightarrow 0$ as $M \rightarrow \infty$ if φ lies in this interval, but not otherwise. That this is the case is shown in Figure 8.

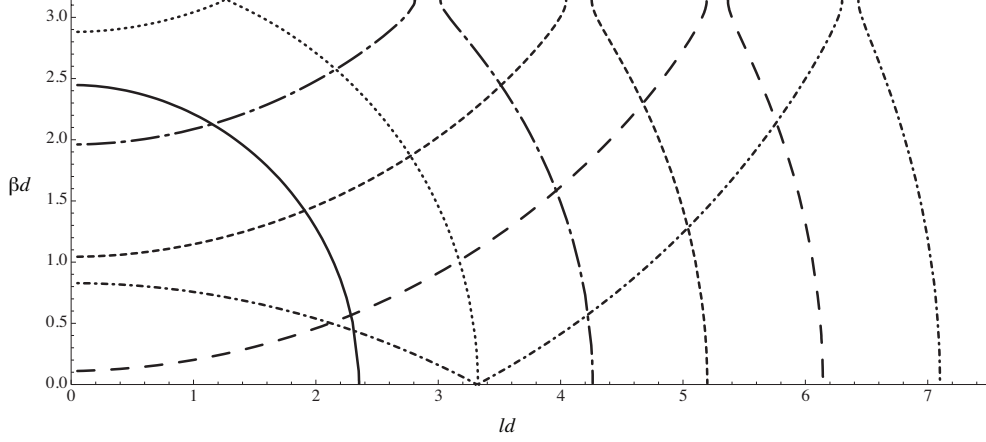


FIGURE 7. Dispersion curves showing βd as a function of ld for various values of Kd when $f/d = 0.5$, $a/d = 0.25$. From ‘left to right’ we have $Kd = 2, 3, 4, 5, 6, 7$.

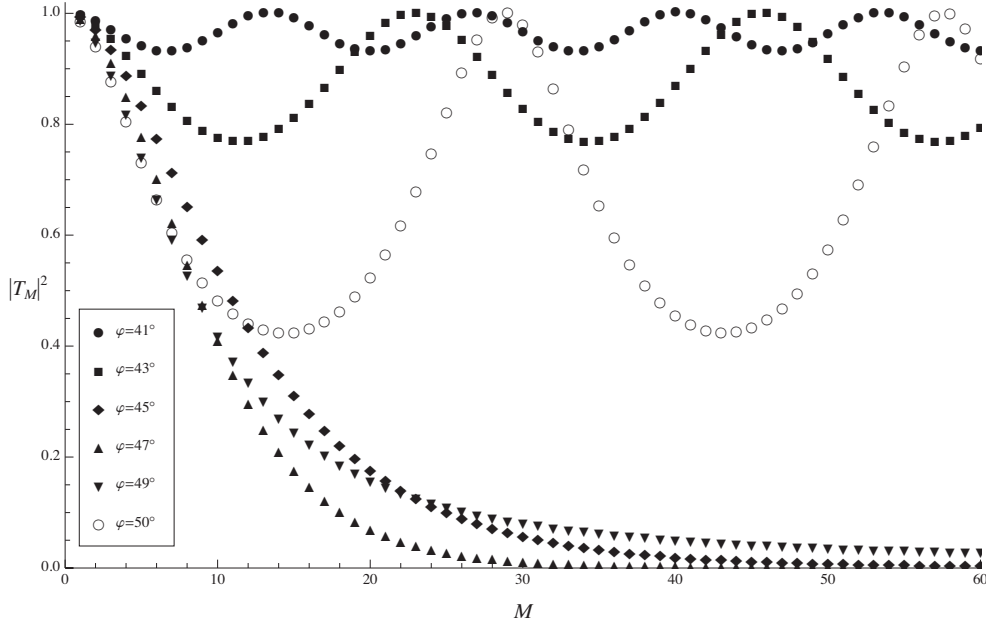


FIGURE 8. Transmitted energy as a function of the number of cylinders in the array for $Ka = 1$, $f/d = 0.5$, $a/d = 0.25$.

6. Conclusion

The existence of separate pass and stop bands associated with wave propagation through periodic media is a common phenomenon, though it appears to have received rather little attention in the context of the full linear theory of water waves. In this article we have focussed on water waves propagating over periodic arrays of submerged horizontal circular cylinders in deep water, considering waves travelling at an oblique angle as well as normal to the cylinder axes.

The fact that submerged circular cylinders are weak reflectors makes this configuration particularly interesting. In a sense they are the weakest of all reflectors since the reflection

coefficient for a single cylinder (with normal incidence) is zero for all frequencies. Thus it is not obvious that there will be any stop bands. The fact that there are, as we have shown, indicates that the evanescent interactions between the cylinders must play a crucial role in their creation and heuristic arguments that are often put forward to explain Bragg resonance are inadequate in this case. A zero reflection coefficient does not imply that the waves are unaffected by the cylinder and it is well known that there is a phase shift as the wave propagates over such a structure. The existence of this phase shift means that the standard Bragg law, which can be thought of as a first approximation to the location of a band gap, needs to be modified.

The multipole expansion technique has been used to solve both the case of an infinite array and a finite array of cylinders. This approach appears to work particularly well for the infinite array case and normal incidence since some of the infinite sums that are required can be evaluated in closed form. Numerical results show that band gaps do exist for both normally propagating and oblique waves and we have shown how this corresponds to cases in which the energy which is transmitted across a finite array tends to zero as the size of the array increases.

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