Probabilistic Representation of Weak Solutions of Partial Differential Equations with Polynomial Growth Coefficients

Qi Zhang [a], Huaizhong Zhao [b]

[a] School of Mathematical Sciences, Fudan University, Shanghai, 200433, China.
[b] Department of Mathematical Sciences, Loughborough University, Loughborough, LE11 3TU, UK.
Emails: qzh@fudan.edu.cn; H.Zhao@lboro.ac.uk

Summary. In this paper we develop a new weak convergence and compact embedding method to study the existence and uniqueness of the $L_\rho^2(\mathbb{R}^d;\mathbb{R}^1)\otimes L_\rho^2(\mathbb{R}^d;\mathbb{R}^d)$ valued solution of backward stochastic differential equations with p-growth coefficients. Then we establish the probabilistic representation of the weak solution of PDEs with p-growth coefficients via corresponding BSDEs.

Keywords: PDEs with polynomial growth coefficients, generalized Feynman-Kac formula, probabilistic representation of weak solutions, backward stochastic differential equations, weak convergence, compact embedding.

AMS 2000 subject classifications: 60H10, 60H30, 35K55.

1 Introduction

In this paper, we study the probabilistic representation of the weak solution of a class of parabolic partial differential equations (PDEs) on $\mathbb{R}^d$ with p-growth coefficients

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x) = \mathscr{L}v(t,x) + f\big(x, v(t,x), (\sigma^*\nabla v)(s,x)\big), \quad 0\leq t\leq T, \\ v(0,x) = h(x), \end{cases} \tag{1.1}$$

by the solution of the corresponding backward stochastic differential equations (BSDEs) in ρ-weighted L^2 space. Here $\mathscr{L}$ is a second order differential operator

$$\mathscr{L} = \frac{1}{2}\sum_{i,j=1}^d a_{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j} + \sum_{i=1}^d b_i(x)\frac{\partial}{\partial x_i}, \tag{1.2}$$

$(a_{ij}(x))$ is a symmetric matrix with a decomposition $(a_{ij}(x)) = (\sigma_{ij}(x))(\sigma_{ij}(x))^*$, $f:(x,y,z)\mapsto f(x,y,z)$ is a function of polynomial growth in y and Lipschitz continuous in z. Many partial differential equations arising in physics, engineering and biology have polynomial growth non-linear terms e.g. KPP-Fisher equations, Allen-Cahn equations and Ginzburg-Landau equations. The representation provides an important connection between stochastic flows generated by $\mathscr{L}$ and the weak solutions of PDEs possibly with polynomial growth coefficients. In connection with the classical solutions of the linear parabolic PDEs, the well-known Feynman-Kac formula provides the probabilistic representation for them and originated many important developments (Feynman [8], Kac [12]). An alternative probabilistic representation using only the values of a finite (random) set of times to the linear heat equations was obtained recently by Dalang, Mueller and Tribe [4]. This idea made it possible for them to obtain corresponding formula for a wide class of linear PDEs such as some wave equations with potentials. The Feynman-Kac formula has played important roles in problems such as the large deviation theory of Donsker and Varadhan [6], Wentzell and Freidlin [23], small time asymptotics of heat kernel and its logarithmic derivatives, in particular on Riemannian manifolds (Elworthy [10], Malliavin and Stroock [16]). The Feynman-Kac formula has been extended and used to quasi-linear parabolic type partial differential equations, especially, in the study of the generalized KPP equations using the large deviation theory method by Freidlin [9], using the semi-classical probabilistic method by Elworthy, Truman and Zhao [7]. The study of the quasi-linear parabolic type PDE is based on an equation of the Feynman-Kac type integration of stochastic functionals. The approach of the backward stochastic differential equations, pioneered by Pardoux and Peng [19], [20] originally, provided an alternative approach to the classical solution of the parabolic type PDEs, when the coefficients of the PDE are sufficiently regular and Lipschitz continuous. This was extended to the viscosity solution of a large class of partial differential equations and BSDEs. They include the linear growth case considered by Lepeltier and San Martin [15], the quadratic coefficients (in z) considered by Kobylanski [13], Briand and Hu [3], and the polynomial growth coefficients in Pardoux [18]. The solution of the BSDEs in above cases gives the probabilistic representation of the classical or viscosity solution of the PDEs as a generalization to the Feynman-Kac formula. Applications of BSDEs have been found in some problems such as a model in mathematics of finance (El Karoui, Peng and Quenez [11]), as an efficient method for constructing Γ-martingales on Riemannian manifolds (Darling [5]), and as an intrinsic tool to construct the pathwise stationary solution for stochastic PDEs (Zhang and Zhao [24], [25]).

The Feynman-Kac approach to a Sobolev or L^2 space valued weak solution of PDEs has been concentrated mainly on linear problems. Many important progress has been made e.g. in quantum field theory (see [22]). The probabilistic approach to the weak solution of quasi-linear PDEs stayed behind. Regularity of the solutions, even in the sense of weak derivative, was not given in Freidlin's probabilistic approach of generalized solution formally represented by the Feynman-Kac formula ([9]). The BSDEs start to show some usefulness in this aspect, when the coefficients are of Lipschitz continuous in the space $L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)\otimes L^2_\rho(\mathbb{R}^d;\mathbb{R}^d)$ or of linear growth, and monotone, from the work of Barles and Lesigne [2], Bally and Matoussi [1], Zhang and Zhao [24], [25]. The objective of this paper is to move away from the assumption of the linear growth of f and from considering the classical or viscosity solution of PDEs to establish the probabilistic representation for the weak solution of such polynomial growth PDEs. Although the connection of BSDEs with the viscosity solution for the cases of quadratic and polynomial growth has been obtained in [13], [18] respectively, the existing methods in the study of BSDEs for finding the solution of the BSDEs in $L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)\otimes L^2_\rho(\mathbb{R}^d;\mathbb{R}^d)$ are not adequate to solve the problem of the weak solution of BSDEs with p-growth coefficients. The fixed point method in $M^2([t,T];L^2_\rho(\mathbb{R}^d;\mathbb{R}^1))\otimes M^2([t,T];L^2_\rho(\mathbb{R}^d;\mathbb{R}^d))$, which is equivalent to finding a strongly convergent sequence in the same space, seems difficult to work for the problem with p-growth coefficients. It is also inadequate to use a combination of the weak convergence in finite dimensional space developed by Pardoux [18] and the weak solution method developed by Bally and Matoussi [1], Zhang and Zhao [24], [25] to solve this problem. We need to introduce some new ideas to the study of BSDEs. The progress of this problem was made when we realized that, in addition to the method of Zhang and Zhao ([24], [25]), as well as the standard approach using Alaoglu lemma to find a weakly convergent sequence (Y^n, Z^n), we can use the equivalence of norm principle and Rellich-Kondrachov Compactness Theorem to get a strongly convergent sequence Y^n. Our recent result on the $S^2([t,T], L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)) \otimes M^2([t,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^d))$ valued solution of BDSDEs with nonlipschitz linear growth coefficients made it possible for us to study the BSDEs in $S^{2p}([t,T], L^{2p}_\rho(\mathbb{R}^d;\mathbb{R}^1)) \otimes M^2([t,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^d))$ with polynomial growth coefficients, even without assuming f being locally Lipschitz continuous in y. Of course, we need to assume the monotonicity condition of f in y. Moreover, it is also an essential step to prove the strong convergence of Z^n in $M^2([t,T], L^2_\rho(\mathbb{R}^d;\mathbb{R}^d))$ from the result of the strong convergence of Y^n and Itô's formula. The weak convergence and compact embedding method has been used in the study of PDEs. However, as far as we know, to use this kind of argument to the study of BSDE, this paper is the first time in literature. The equivalence of norm principle and very careful probabilistic (measure theoretical) and analytic arguments including localization made it work in the probabilistic context. However, the probabilistic case is a lot more complicated than the deterministic PDEs case as we need to work on the space $\Omega \otimes [0,T] \otimes \mathbb{R}^d$ and solve the equation with probabilistic one, instead only work on $[0,T] \otimes \mathbb{R}^d$ in the deterministic PDEs case. The probabilistic representation can be regarded as a generalized Feynman-Kac formula to the weak solution of the PDEs with p-growth coefficients and is new in literature. We believe our method will be useful to other types of BSDEs and PDEs as well.

After this paper was completed, we were informed the paper Matoussi and Xu [17]. But we would like to point out what we have proved as well as our methods are different. Notice the convergence $(Y^{t,x,n}_s, Z^{t,x,n}_s)$ is only a weak convergence along a subsequence according to the Alaoglu lemma. If one considers weak convergence in $M^2([t,T],\mathbb{R}^1) \otimes M^2([t,T];\mathbb{R}^d))$, which worked well in Pardoux [18] for the case of viscosity solutions of the PDEs, then each weak convergence is for a fixed x, and the choice of subsequence may depend on x. However, this will cause serious problems when one considers weak solutions. Our approach to avoid this essential difficulty is to find a subsequence of the weak convergence in the space $M^2([t,T], L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)) \otimes M^2([t,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^d))$. The whole point and major difficulty of this approach are to pass the limit term by term in the approximating equation to the desired limit. This is achieved in our paper by obtaining a strong convergent subsequence of $(Y^{t,x,n}_s, Z^{t,x,n}_s)$ in $M^2([t,T], L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)) \otimes M^2([t,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^d))$ using the Rellich-Kondrachov compactness theorem and generalized equivalence of norm principle as we have already mentioned.

2 The main results

In this paper, we study the weak solutions of a class of parabolic PDEs with p-growth coefficients, their corresponding backward stochastic differential equations (BSDEs) in a Hilbert space (ρ-weighted L^2 space) and the probabilistic representation of the weak solutions of (1.1) by using the solutions of BSDEs. We start from the following SDE:

$$X^{t,x}_s = x + \int_t^s b(X^{t,x}_r)dr + \int_t^s \sigma(X^{t,x}_r)dW_r, \quad s \geq t, \tag{2.1}$$

where W is a $\mathbb{R}^d$ Brownian motion on a probability space $(\Omega, \mathscr{F}, P)$, and $b : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ are measurable. We consider a slightly more general BSDEs by allowing f depending on time explicitly:

$$Y_s^{t,x} = h(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr - \int_s^T \langle Z_r^{t,x}, dW_r\rangle, \tag{2.2}$$

where $f : [0,T]\times\mathbb{R}^d\times\mathbb{R}^1\times\mathbb{R}^d \to \mathbb{R}^1$ and $h : \mathbb{R}^d \to \mathbb{R}^1$ are measurable. More conditions on b, σ, f are needed and will be specified later. The Hilbert space $L^2_\rho(\mathbb{R}^d;\mathbb{R}^k)$ is the space containing all Borel measurable functions l: $\mathbb{R}^d \to \mathbb{R}^k$ such that $\int_{\mathbb{R}^d} < l(x), l(x) > \rho^{-1}(x)dx < \infty$, with the inner product

$$\langle u_1, u_2\rangle = \int_{\mathbb{R}^d} < u_1(x), u_2(x) > \rho^{-1}(x)dx,$$

where $\rho(x) = (1 + |x|)^q$, $q > d$, is a weight function. The Banach space $L^{2p}_\rho(\mathbb{R}^d;\mathbb{R}^1)$ is the space containing all Borel measurable functions l: $\mathbb{R}^d \to \mathbb{R}^1$ such that $\int_{\mathbb{R}^d} l^{2p}(x)\rho^{-1}(x)dx < \infty$ with the norm $||l||_{L^{2p}_\rho(R^d)} = (\int_{\mathbb{R}^d} l^{2p}(x)\rho^{-1}(x)dx)^{\frac{1}{2p}}$. It is easy to see that $\rho(x) : \mathbb{R}^d \longrightarrow \mathbb{R}^1$ is a continuous positive function satisfying $\int_{\mathbb{R}^d} \rho^{-1}(x)dx < \infty$. Note that we can consider more general ρ which satisfies the above condition and conditions in [1] and all the results of this paper still hold. For $k \geq 0$, we denote by C_b^k the set of C^k-functions whose partial derivatives of order less than or equal to k are bounded and by H^k_ρ the ρ-weighted Sobolev space (See e.g. [1]). Now we assume the following conditions for the coefficients in SDE (2.1) and BSDE (2.2):

(H.1). For a given $p \geq 1$, $\int_{\mathbb{R}^d} |h(x)|^{2p}\rho^{-1}(x)dx < \infty$.

(H.2). There exists a constant $C \geq 0$ and a function f_0 with $\int_0^T \int_{\mathbb{R}^d} |f_0(s,x)|^{2p}\rho^{-1}(x)dxds < \infty$ s.t. $|f(s,x,y,z)| \leq C(|f_0(s,x)| + |y|^p + |z|)$, where p is the same as in (H.1).

(H.3). There exists a constant $\mu \in \mathbb{R}^1$ s.t. for any $s \in [0,T]$, $y_1, y_2 \in \mathbb{R}^1$, $x, z \in \mathbb{R}^d$,

$$(y_1 - y_2)\big(f(s,x,y_1,z) - f(s,x,y_2,z)\big) \leq \mu|y_1 - y_2|^2.$$

(H.4). The function $(y,z) \to f(s,x,y,z)$ is continuous and $z \to f(s,x,y,z)$ is globally Lipschitz continuous with Lipschitz constant $L \geq 0$, i.e. for any $s \in [0,T]$, $y \in \mathbb{R}^1$, $x, z_1, z_2 \in \mathbb{R}^d$,

$$|f(s,x,y,z_1) - f(s,x,y,z_2)| \leq L|z_1 - z_2|.$$

(H.5). The diffusion coefficients $b \in C_b^2(\mathbb{R}^d;\mathbb{R}^d)$, $\sigma \in C_b^3(\mathbb{R}^d;\mathbb{R}^d\times\mathbb{R}^d)$ and σ satisfies the uniform ellipticity condition, i.e. there exists a constant $D > 0$ s.t. $\xi^*(\sigma\sigma^*)(x)\xi \geq D\xi^*\xi$ for any $\xi \in \mathbb{R}^d$.

It is easy to see that for a.e. $x \in \mathbb{R}^d$, $(Y_s^{t,x}, Z_s^{t,x})$ solves BSDE (2.2) if and only if $(\tilde{Y}_s^{t,x}, \tilde{Z}_s^{t,x}) = (\mathrm{e}^{\mu s}Y_s^{t,x}, \mathrm{e}^{\mu s}Z_s^{t,x})$ solves the following BSDE:

$$\tilde{Y}_s^{t,x} = \mathrm{e}^{\mu T}h(X_T^{t,x}) + \int_s^T \tilde{f}(r, X_r^{t,x}, \tilde{Y}_r^{t,x}, \tilde{Z}_r^{t,x})dr - \int_s^T \langle \tilde{Z}_r^{t,x}, dW_r\rangle, \tag{2.3}$$

where $\tilde{f}(r,x,y,z) = \mathrm{e}^{\mu r}f(r,x,\mathrm{e}^{-\mu r}y, \mathrm{e}^{-\mu r}z) - \mu y$. We can verify that $\tilde{f}$ satisfies Conditions (H.2), (H.3) and (H.4). But, by Condition (H.3), for $y_1, y_2 \in \mathbb{R}^1$, and $x, z \in \mathbb{R}^1$,

$$
\begin{aligned}
&(y_1-y_2)\big(\tilde f(s,x,y_1,z)-\tilde f(s,x,y_2,z)\big)\\
&=\mathrm{e}^{2\mu s}(\mathrm{e}^{-\mu s}y_1-\mathrm{e}^{-\mu s}y_2)\big(f(s,x,\mathrm{e}^{-\mu s}y_1,\mathrm{e}^{-\mu s}z)-f(s,x,\mathrm{e}^{-\mu s}y_2,\mathrm{e}^{-\mu s}z)\big)-\mu(y_1-y_2)(y_1-y_2)\\
&\leq \mu\mathrm{e}^{2\mu s}|\mathrm{e}^{-\mu s}y_1-\mathrm{e}^{-\mu s}y_2|^2-\mu|y_1-y_2|^2=0.
\end{aligned}
$$

Now we give the definition for the solution of BSDE (2.2) in the ρ-weighted L^2 space. First define the space for the solution $(Y_\cdot^{t,\cdot},Z_\cdot^{t,\cdot})$. We denote by $\mathcal{N}$ the class of P-null sets of $\mathscr{F}$ and let $\mathscr{F}_t\triangleq\mathscr{F}_t^W\bigvee\mathcal{N}$, for $0\leq t\leq T$. We recall some definitions.

Definition 2.1 *(Definitions 2.2 in [24]) Let $\mathbb{S}$ be a Banach space with norm $\|\cdot\|_{\mathbb{S}}$ and Borel σ-field $\mathscr{S}$ and $q\geq 2$ be a real number. We denote by $M^q([t,T];\mathbb{S})$ the set of $\mathscr{B}([t,T])\otimes\mathscr{F}/\mathscr{S}$ measurable random processes $\{\phi(s)\}_{t\leq s\leq T}$ with values in $\mathbb{S}$ satisfying*

(i) $\phi(s):\Omega\to\mathbb{S}$ is $\mathscr{F}_s$ measurable for $t\leq s\leq T$;
(ii) $E[\int_t^T\|\phi(s)\|_{\mathbb{S}}^q ds]<\infty$.

Also we denote by $S^q([t,T];\mathbb{S})$ the set of $\mathscr{B}([t,T])\otimes\mathscr{F}/\mathscr{S}$ measurable random processes $\{\psi(s)\}_{t\leq s\leq T}$ with values in $\mathbb{S}$ satisfying

(i) $\psi(s):\Omega\to\mathbb{S}$ is $\mathscr{F}_s$ measurable for $t\leq s\leq T$ and $\psi(\cdot,\omega)$ is continuous P-a.s.;
(ii) $E[\sup_{t\leq s\leq T}\|\psi(s)\|_{\mathbb{S}}^q]<\infty$.

Definition 2.2 *(Definitions 3.1 in [24]) A pair of processes $(Y_s^{t,x},Z_s^{t,x})$ is called a solution of BSDE (2.2) if $(Y_\cdot^{t,\cdot},Z_\cdot^{t,\cdot})\in S^{2p}([t,T];L_\rho^{2p}(\mathbb{R}^d;\mathbb{R}^1))\bigotimes M^2([t,T];L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$ and $(Y_s^{t,x},Z_s^{t,x})$ satisfies (2.2) for a.e. x, with probability one.*

Since $(Y_\cdot^{t,\cdot},Z_\cdot^{t,\cdot})\in S^{2p}([t,T];L_\rho^{2p}(\mathbb{R}^d;\mathbb{R}^1))\otimes M^2([t,T];L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$ if and only if $(\tilde Y_\cdot^{t,\cdot},\tilde Z_\cdot^{t,\cdot})\in S^{2p}([t,T];L_\rho^{2p}(\mathbb{R}^d;\mathbb{R}^1))\otimes M^2([t,T];L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$, so we claim $(Y_s^{t,x},Z_s^{t,x})$ is the solution of BSDE (2.2) in the ρ-weighted L^2 space if and only if $(\tilde Y_s^{t,x},\tilde Z_s^{t,x})$ is the solution of BSDE (2.3) in ρ-weighted L^2 space. Therefore we can replace, without losing any generality, Condition (A.4) by

(H.3)*. For any $s\in[0,T]$, $y_1,y_2\in\mathbb{R}^1$, $x,z\in\mathbb{R}^d$,

$$
(y_1-y_2)\big(f(s,x,y_1,z)-f(s,x,y_2,z)\big)\leq 0.
$$

The main purpose of this paper is to prove the following two theorems. The first one is about the existence and uniqueness of solutions to BSDE (2.2):

Theorem 2.3 *Under Conditions (H.1), (H.2), (H.3)*, (H.4) and (H.5), BSDE (2.2) has a unique solution $(Y_\cdot^{t,\cdot},Z_\cdot^{t,\cdot})\in S^{2p}([t,T];L_\rho^{2p}(\mathbb{R}^d;\mathbb{R}^1))\otimes M^2([t,T];L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$.*

BSDE (2.2) corresponds to the following PDE with p-growth coefficients:

$$
\begin{cases}
\dfrac{\partial u}{\partial t}(t,x)=-\mathscr{L}u(t,x)-f\big(t,x,u(t,x),(\sigma^*\nabla u)(s,x)\big), & 0\leq t\leq T,\\
u(T,x)=h(x).
\end{cases}
\tag{2.4}
$$

The other main theorem is the probabilistic representation of PDE (2.4) in the ρ-weighted L^2 space through its corresponding BSDE:

Theorem 2.4 *Define* $u(t,x) = Y_t^{t,x}$, *where* $(Y_s^{t,x}, Z_s^{t,x})$ *is the solution of BSDE (2.2) under Conditions (H.1), (H.2), (H.3)*, (H.4) and (H.5), then* $u(t,x)$ *is the unique weak solution of PDE (2.4). Moreover,*

$$u(s, X_s^{t,x}) = Y_s^{t,x},\ (\sigma^*\nabla u)(s, X_s^{t,x}) = Z_s^{t,x} \text{ for a.a. } s \in [t,T],\ x \in \mathbb{R}^d \text{ a.s.}$$

Noticing f is of p-growth on y, we recall the definition for the weak solution of PDE (2.4):

Definition 2.5 *Function* u *is called the weak solution of PDE (2.4) if* $(u, \sigma^*\nabla u) \in L^{2p}([0,T]; L_\rho^{2p}(\mathbb{R}^d;\mathbb{R}^1)) \bigotimes L^2([0,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$ *and for an arbitrary* $\varphi \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^1)$,

$$\begin{aligned}
&\int_{\mathbb{R}^d} u(t,x)\varphi(x)dx - \int_{\mathbb{R}^d} u(T,x)\varphi(x)dx - \frac{1}{2}\int_t^T\int_{\mathbb{R}^d} \big((\sigma^*\nabla u)(s,x)\big)^*(\sigma^*\nabla\varphi)(x)dxds \\
&\quad - \int_t^T\int_{\mathbb{R}^d} u(s,x)div\big((b-\tilde{A})\varphi\big)(x)dxds \\
&= \int_t^T\int_{\mathbb{R}^d} f\big(s,x,u(s,x),(\sigma^*\nabla u)(s,x)\big)\varphi(x)dxds. \qquad (2.5)
\end{aligned}$$

Here $\tilde{A}_j \triangleq \frac{1}{2}\sum_{i=1}^d \frac{\partial a_{ij}(x)}{\partial x_i}$, *and* $\tilde{A} = (\tilde{A}_1, \tilde{A}_2, \cdots, \tilde{A}_d)^*$.

We give the proofs of these two theorems in the latter sections.

In Sections 3-5, by making use of truncated BSDEs, we first deal with BSDE (2.2). To prove BSDE (2.2) has a unique solution, we use the Alaoglu lemma to derive a weak convergence sequence in Section 3 and further use the equivalence of norm principle and Rellich-Kondrachov Compactness Theorem to get a strong convergence sequence in Section 4. Then we complete the proofs of Theorem 2.3 in Section 5 and consider the corresponding PDE (2.4) to obtain Theorem 2.4 in Section 6 which gives the probabilistic representation to the weak solution of PDE (2.4).

Remark 2.6 *Let* u *be the weak solution of PDE (2.4) with coefficient* $f\big(x,u,(\sigma^*\nabla u)\big)$ *which is independent of* t, *we can see easily that* $v(t) \triangleq u(T-t)$ *is the unique weak solution of PDE (1.1).*

3 The weak convergence

Assume f satisfies Conditions (H.2), (H.3)* and (H.4). We first use a standard cut-off technique to study a sequence of BSDEs with nonlinear function f_n satisfying the linear growth condition on y. The $S^{2p}([t,T]; L_\rho^{2p}(\mathbb{R}^d;\mathbb{R}^1)) \otimes M^2([t,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$ valued solution for this kind equation was studied in [25]. For this, we define for each $n \in N$

$$f_n(s,x,y,z) = f\big(s,x,\mathit{\Pi}_n(y),z\big), \qquad (3.1)$$

where $\mathit{\Pi}_n(y) = \frac{\inf(n,|y|)}{|y|}y$. Then $f_n : [0,T]\times\mathbb{R}^d\times\mathbb{R}^1\times\mathbb{R}^d \to \mathbb{R}^1$ satisfies

(H.2)$'$. For any $s \in [0,T]$, $y \in \mathbb{R}^1$, $x, z \in \mathbb{R}^d$ and the constant C given in (H.2),

$$|f_n(s,x,y,z)| \leq C(|f_0(s,x)| + |n|^p + |z|).$$

(H.3)$'$. For any $s \in [0,T]$, $y_1, y_2 \in \mathbb{R}^1$, $x \in \mathbb{R}^d$,

$$(y_1 - y_2)\big(f_n(s,x,y_1,z) - f_n(s,x,y_2,z)\big) \leq 0.$$

(H.4)$'$. The function $(y,z) \to f_n(s,x,y,z)$ is continuous, and for any $s \in [0,T]$, $y \in \mathbb{R}^1$, $x, z_1, z_2 \in \mathbb{R}^d$ and the constant L given in (H.4),

$$|f_n(s,x,y,z_1) - f_n(s,x,y,z_2)| \leq L|z_1 - z_2|.$$

To see (H.3)$'$, if $\Pi_n(y_1) = \Pi_n(y_2)$, it is obvious; if $\Pi_n(y_1) \neq \Pi_n(y_2)$, then

$$\begin{aligned}&(y_1 - y_2)\big(f_n(s,x,y_1,z) - f_n(s,x,y_2,z)\big)\\ &= (\Pi_n(y_1) - \Pi_n(y_2))\big(f(s,x,\Pi_n(y_1),z) - f(s,x,\Pi_n(y_2),z)\big)\frac{y_1 - y_2}{\Pi_n(y_1) - \Pi_n(y_2)} \leq 0.\end{aligned}$$

We then study the following BSDE with the global Lipschitz coefficient f_n:

$$Y_s^{t,x,n} = h(X_T^{t,x}) + \int_s^T f_n(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n})dr - \int_s^T \langle Z_r^{t,x,n}, dW_r\rangle. \tag{3.2}$$

Notice that under the conditions of Theorem 2.3, the coefficients h and f_n satisfy Conditions (H.1), (H.2)$'$ and (H.4)$'$. Hence by Theorems 2.2 and 2.3 in [25], we have the following proposition:

Proposition 3.1 *([25]) Under the conditions of Theorem 2.3, for f_n defined in (3.1), BSDE (3.2) has a unique solution $(Y_s^{t,x,n}, Z_s^{t,x,n}) \in S^2([t,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)) \otimes M^2([t,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^d))$. If we define $Y_t^{t,x,n} = u_n(t,x)$, then $u_n(t,x)$ is the unique weak solution of the following PDE*

$$\begin{cases} \dfrac{\partial u_n}{\partial t}(t,x) = -\mathscr{L}u_n(t,x) - f_n\big(t,x,u_n(t,x),(\sigma^*\nabla u)(t,x)\big), \quad 0 \leq t \leq T,\\ u_n(T,x) = h(x). \end{cases} \tag{3.3}$$

Moreover,

$$u_n(s, X_s^{t,x}) = Y_s^{t,x,n},\ (\sigma^*\nabla u_n)(s, X_s^{t,x}) = Z_s^{t,x,n} \text{ for a.a. } s \in [t,T],\ x \in \mathbb{R}^d \text{ a.s.}$$

The key is to pass the limits in (3.2) and (3.3) in some desired sense. For this we need some estimates that go beyond those in [24] and [25]. Before we derive some useful estimations to the solution of BSDEs (3.2), we give the generalized equivalence of norm principle which is an extension of equivalence of norm principle given in [14], [2], [1] to the cases when φ and Ψ are random.

Lemma 3.2 *(generalized equivalence of norm principle [24]) Let ρ be the weight function defined at the beginning of Section 1 and X be a diffusion process defined in (2.1). If $s \in [t,T]$, $\varphi : \Omega \times \mathbb{R}^d \to \mathbb{R}^1$ is independent of the σ-field $\sigma\{W_r - W_t,\ t \leq r \leq s\}$ and $\varphi\rho^{-1} \in L^1(\Omega \otimes \mathbb{R}^d)$, then there exist two constants $c > 0$ and $C > 0$ such that*

$$cE[\int_{\mathbb{R}^d} |\varphi(x)|\rho^{-1}(x)dx] \leq E[\int_{\mathbb{R}^d} |\varphi(X_s^{t,x})|\rho^{-1}(x)dx] \leq CE[\int_{\mathbb{R}^d} |\varphi(x)|\rho^{-1}(x)dx].$$

Moreover if $\Psi : \Omega \times [t,T] \times \mathbb{R}^d \to \mathbb{R}^1$, $\Psi(s,\cdot)$ is independent of $\mathscr{F}_{t,s}^W$ and $\Psi\rho^{-1} \in L^1(\Omega \otimes [t,T] \otimes \mathbb{R}^d)$, then

$$cE[\int_t^T\int_{\mathbb{R}^d}|\Psi(s,x)|\rho^{-1}(x)dxds]\leq E[\int_t^T\int_{\mathbb{R}^d}|\Psi(s,X_s^{t,x})|\rho^{-1}(x)dxds]$$
$$\leq CE[\int_t^T\int_{\mathbb{R}^d}|\Psi(s,x)|\rho^{-1}(x)dxds].$$

First we deduce a useful estimate.

Lemma 3.3 *Under Conditions (H.1), (H.2), (H.3)*, (H.4) and (H.5), if* $(Y_\cdot^{t,\cdot,n},Z_\cdot^{t,\cdot,n})$ *is the solution of BSDE (3.2), then we have*

$$E[\int_t^T\sup_n\int_{\mathbb{R}^d}|Y_s^{t,x,n}|^{2p}\rho^{-1}(x)dxds]+\sup_n E[\int_t^T\int_{\mathbb{R}^d}|Y_s^{t,x,n}|^{2p-2}|Z_s^{t,x,n}|^2\rho^{-1}(x)dxds]<\infty.$$

Proof. For M, $N>0$ and $m\geq 2$, define

$$\psi_M(y)=y^2I_{\{-M\leq y<M\}}+M(2y-M)I_{\{y\geq M\}}-M(2y+M)I_{\{y<-M\}}$$

and

$$\varphi_{N,m}(y)=y^{\frac{m}{2}}I_{\{0\leq y<N\}}+N^{\frac{m-2}{2}}(\frac{m}{2}y-\frac{m-2}{2}N)I_{\{y\geq N\}}.$$

Applying Itô's formula to $\mathrm{e}^{Kr}\varphi_{N,m}\big(\psi_M(Y_r^{t,x,n})\big)$ for a.e. $x\in\mathbb{R}^d$, we have

$$\begin{aligned}
&\mathrm{e}^{Ks}\varphi_{N,m}\big(\psi_M(Y_s^{t,x,n})\big)+K\int_s^T\mathrm{e}^{Kr}\varphi_{N,m}\big(\psi_M(Y_r^{t,x,n})\big)dr\\
&+\frac{1}{2}\int_s^T\mathrm{e}^{Kr}\varphi_{N,m}^{''}\big(\psi_M(Y_r^{t,x,n})\big)|\psi_M^{'}(Y_r^{t,x,n})|^2|Z_r^{t,x,n}|^2dr\\
&+\int_s^T\mathrm{e}^{Kr}\varphi_{N,m}^{'}\big(\psi_M(Y_r^{t,x,n})\big)I_{\{-M\leq Y_r^{t,x,n}<M\}}|Z_r^{t,x,n}|^2dr\\
=&\,\mathrm{e}^{KT}\varphi_{N,m}\big(\psi_M(h(X_T^{t,x}))\big)+\int_s^T\mathrm{e}^{Kr}\varphi_{N,m}^{'}\big(\psi_M(Y_r^{t,x,n})\big)\psi_M^{'}(Y_r^{t,x,n})f_n(r,X_r^{t,x},Y_r^{t,x,n},Z_r^{t,x,n})dr\\
&-\int_s^T\langle\mathrm{e}^{Kr}\varphi_{N,m}^{'}\big(\psi_M(Y_r^{t,x,n})\big)\psi_M^{'}(Y_r^{t,x,n})Z_r^{t,x,n},dW_r\rangle. \qquad (3.4)
\end{aligned}$$

From [24], we note first $(Y_\cdot^{t,\cdot,n},Z_\cdot^{t,\cdot,n})\in S^2([0,T];L_\rho^2(\mathbb{R}^d;\mathbb{R}^1))\bigotimes M^2([0,T];L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$. Also it is obvious that $\varphi_{N,m}^{'}\big(\psi_M(Y_r^{t,x,n})\big)\psi_M^{'}(Y_r^{t,x,n})$ is bounded, hence we can use the stochastic Fubini theorem and take the conditional expectation w.r.t. $\mathscr{F}_s$. Note that the stochastic integral has zero conditional expectation. So if we define $\frac{\psi_M^{'}(y)}{y}=2$ when $y=0$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d}\mathrm{e}^{Ks}\varphi_{N,m}\big(\psi_M(Y_s^{t,x,n})\big)\rho^{-1}(x)dx+E[K\int_s^T\int_{\mathbb{R}^d}\mathrm{e}^{Kr}\varphi_{N,m}\big(\psi_M(Y_r^{t,x,n})\big)\rho^{-1}(x)dxdr|\mathscr{F}_s]\\
&+\frac{1}{2}E[\int_s^T\int_{\mathbb{R}^d}\mathrm{e}^{Kr}\varphi_{N,m}^{''}\big(\psi_M(Y_r^{t,x,n})\big)|\psi_M^{'}(Y_r^{t,x,n})|^2|Z_r^{t,x,n}|^2\rho^{-1}(x)dxdr|\mathscr{F}_s]\\
&+E[\int_s^T\int_{\mathbb{R}^d}\mathrm{e}^{Kr}\varphi_{N,m}^{'}\big(\psi_M(Y_r^{t,x,n})\big)I_{\{-M\leq Y_r^{t,x,n}<M\}}|Z_r^{t,x,n}|^2\rho^{-1}(x)dxdr|\mathscr{F}_s]\\
=&\,E[\int_{\mathbb{R}^d}\mathrm{e}^{KT}\varphi_{N,m}\big(\psi_M(h(X_T^{t,x}))\big)\rho^{-1}(x)dx|\mathscr{F}_s]
\end{aligned}$$

$$
\begin{aligned}
&+E[\int_s^T\int_{\mathbb{R}^d}\mathrm{e}^{Kr}\varphi'_{N,m}\big(\psi_M(Y_r^{t,x,n})\big)\psi'_M(Y_r^{t,x,n})f_n(r,X_r^{t,x},Y_r^{t,x,n},Z_r^{t,x,n})\rho^{-1}(x)dxdr|\mathscr{F}_s]\\
=\;& E[\int_{\mathbb{R}^d}\mathrm{e}^{KT}\varphi_{N,m}\big(\psi_M(h(X_T^{t,x}))\big)\rho^{-1}(x)dx|\mathscr{F}_s]\\
&+E[\int_s^T\int_{\mathbb{R}^d}\mathrm{e}^{Kr}\varphi'_{N,m}\big(\psi_M(Y_r^{t,x,n})\big)\frac{\psi'_M(Y_r^{t,x,n})}{Y_r^{t,x,n}}Y_r^{t,x,n}\\
&\qquad\times\big(f_n(r,X_r^{t,x},Y_r^{t,x,n},Z_r^{t,x,n})-f_n(r,X_r^{t,x},0,Z_r^{t,x,n})\big)\rho^{-1}(x)dxdr|\mathscr{F}_s]\\
&+E[\int_s^T\int_{\mathbb{R}^d}\mathrm{e}^{Kr}\varphi'_{N,m}\big(\psi_M(Y_r^{t,x,n})\big)\psi'_M(Y_r^{t,x,n})\\
&\qquad\times\big(f_n(r,X_r^{t,x},0,Z_r^{t,x,n})-f_n(r,X_r^{t,x},0,0)\big)\rho^{-1}(x)dxdr|\mathscr{F}_s]\\
&+E[\int_s^T\int_{\mathbb{R}^d}\mathrm{e}^{Kr}\varphi'_{N,m}\big(\psi_M(Y_r^{t,x,n})\big)\psi'_M(Y_r^{t,x,n})f_n(r,X_r^{t,x},0,0)\rho^{-1}(x)dxdr|\mathscr{F}_s]\\
\le\;& E[\int_{\mathbb{R}^d}\mathrm{e}^{KT}\varphi_{N,m}\big(\psi_M(h(X_T^{t,x}))\big)\rho^{-1}(x)dx|\mathscr{F}_s]\\
&+LE[\int_s^T\int_{\mathbb{R}^d}\mathrm{e}^{Kr}|\varphi'_{N,m}\big(\psi_M(Y_r^{t,x,n})\big)||\psi'_M(Y_r^{t,x,n})||Z_r^{t,x,n}|\rho^{-1}(x)dxdr|\mathscr{F}_s]\\
&+E[\int_s^T\int_{\mathbb{R}^d}\mathrm{e}^{Kr}|\varphi'_{N,m}\big(\psi_M(Y_r^{t,x,n})\big)||\psi'_M(Y_r^{t,x,n})||f(r,X_r^{t,x},0,0)|\rho^{-1}(x)dxdr|\mathscr{F}_s].
\end{aligned}
$$

Taking the limit as $M\to\infty$ first, then the limit as $N\to\infty$, by the monotone convergence theorem and Young inequality, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^d}\mathrm{e}^{Ks}|Y_s^{t,x,n}|^m\rho^{-1}(x)dx+KE[\int_s^T\int_{\mathbb{R}^d}\mathrm{e}^{Kr}|Y_r^{t,x,n}|^m\rho^{-1}(x)dxdr|\mathscr{F}_s]\\
&+\frac{m(m-1)}{2}E[\int_s^T\int_{\mathbb{R}^d}\mathrm{e}^{Kr}|Y_r^{t,x,n}|^{m-2}|Z_r^{t,x,n}|^2\rho^{-1}(x)dxdr|\mathscr{F}_s]\\
\le\;& E[\int_{\mathbb{R}^d}\mathrm{e}^{KT}|h(X_T^{t,x})|^m\rho^{-1}(x)dx|\mathscr{F}_s]\\
&+mLE[\int_s^T\int_{\mathbb{R}^d}\mathrm{e}^{Kr}|Y_r^{t,x,n}|^{m-2}|Y_r^{t,x,n}||Z_r^{t,x,n}|\rho^{-1}(x)dxdr|\mathscr{F}_s]\\
&+mE[\int_s^T\int_{\mathbb{R}^d}\mathrm{e}^{Kr}|Y_r^{t,x,n}|^{m-2}|Y_r^{t,x,n}||f(r,X_r^{t,x},0,0)|\rho^{-1}(x)dxdr|\mathscr{F}_s]\\
\le\;& E[\int_{\mathbb{R}^d}\mathrm{e}^{KT}|h(X_T^{t,x})|^m\rho^{-1}(x)dx|\mathscr{F}_s]+m(L^2+1)E[\int_s^T\int_{\mathbb{R}^d}\mathrm{e}^{Kr}|Y_r^{t,x,n}|^m\rho^{-1}(x)dxdr|\mathscr{F}_s]\\
&+\frac{m}{4}E[\int_s^T\int_{\mathbb{R}^d}\mathrm{e}^{Kr}|Y_r^{t,x,n}|^{m-2}|Z_r^{t,x,n}|^2\rho^{-1}(x)dxdr|\mathscr{F}_s]\\
&+\frac{m}{4}\cdot\frac{m-2}{m}E[\int_s^T\int_{\mathbb{R}^d}\mathrm{e}^{Kr}|Y_r^{t,x,n}|^m\rho^{-1}(x)dxdr|\mathscr{F}_s]\\
&+\frac{m}{4}\cdot\frac{2}{m}E[\int_s^T\int_{\mathbb{R}^d}\mathrm{e}^{Kr}|f(r,X_r^{t,x},0,0)|^m\rho^{-1}(x)dxdr|\mathscr{F}_s]. \qquad (3.5)
\end{aligned}
$$

Here and in the following, C_p is a generic constant. Therefore, taking $K>m(L^2+1)+\frac{m-2}{4}$, we have

$$
E[\int_t^T\sup_n\int_{\mathbb{R}^d}|Y_s^{t,x,n}|^m\rho^{-1}(x)dxds]+\sup_n E[\int_t^T\int_{\mathbb{R}^d}|Y_s^{t,x,n}|^{m-2}|Z_s^{t,x,n}|^2\rho^{-1}(x)dxds]
$$

$$\leq C_pE[\int_{\mathbb{R}^d}|h(X_T^{t,x})|^m\rho^{-1}(x)dx]+C_pE[\int_t^T\int_{\mathbb{R}^d}|f_0(s,X_s^{t,x})|^m\rho^{-1}(x)dxds]$$
$$\leq C_p\int_{\mathbb{R}^d}|h(x)|^m\rho^{-1}(x)dx+C_p\int_t^T\int_{\mathbb{R}^d}|f_0(s,x)|^m\rho^{-1}(x)dxds<\infty.$$

In particular, taking $m=2p$, then the lemma follows. ⋄

Taking $m=2$ in the proof of Lemma 3.3, we know

$$E[\int_t^T\sup_n\int_{\mathbb{R}^d}|Y_s^{t,x,n}|^2\rho^{-1}(x)dxds+\sup_n E[\int_t^T\int_{\mathbb{R}^d}|Z_s^{t,x,n}|^2\rho^{-1}(x)dxds]<\infty. \tag{3.6}$$

Also we have

$$\begin{aligned}&\sup_n E[\int_t^T\int_{\mathbb{R}^d}|f_n(s,X_s^{t,x},Y_s^{t,x,n},Z_s^{t,x,n})|^2\rho^{-1}(x)dxds]\\ \leq&\sup_n E[\int_t^T\int_{\mathbb{R}^d}C(|f_0(s,X_s^{t,x})|^2+|Y_s^{t,x,n}|^{2p}+|Z_s^{t,x,n}|^2)\rho^{-1}(x)dxds]<\infty.\end{aligned}$$

The last inequality follows from the equivalence of norms principle and Lemma 3.3. Define $U_s^{t,x,n}=f_n(s,X_s^{t,x},Y_s^{t,x,n},Z_s^{t,x,n})$, $s\geq t$, then

$$\sup_n E[\int_t^T\int_{\mathbb{R}^d}(|Y_s^{t,x,n}|^2+|Z_s^{t,x,n}|^2+|U_s^{t,x,n}|^2)\rho^{-1}(x)dxds]<\infty. \tag{3.7}$$

Therefore by using the Alaoglu lemma, we know that there exists a subsequence, still denoted by $(Y_s^{t,x,n},Z_s^{t,x,n},U_s^{t,x,n})$, s.t. $(Y_s^{t,x,n},Z_s^{t,x,n},U_s^{t,x,n})$ converges weakly to the limit $(Y_s^{t,x},Z_s^{t,x},U_s^{t,x})$ in $L_\rho^2(\Omega\otimes[t,T]\otimes\mathbb{R}^d;\mathbb{R}^1\otimes\mathbb{R}^d\otimes\mathbb{R}^1)$ (or equivalently $L^2(\Omega\otimes[t,T];L_\rho^2(\mathbb{R}^d;\mathbb{R}^1)\otimes L_\rho^2(\mathbb{R}^d;\mathbb{R}^d)\otimes L_\rho^2(\mathbb{R}^d;\mathbb{R}^1))$. Now we take the weak limit in $L_\rho^2(\Omega\otimes[t,T]\otimes\mathbb{R}^d;\mathbb{R}^1))$ to BSDEs (3.2), we can verify that $(Y_s^{t,x},Z_s^{t,x},U_s^{t,x})$ satisfies the following BSDE:

$$Y_s^{t,x}=h(X_T^{t,x})+\int_s^T U_r^{t,x}dr-\int_s^T\langle Z_r^{t,x},dW_r\rangle. \tag{3.8}$$

For this, we will check the weak convergence term by term. The weak convergence to the first term is deduced by the definition of $Y_s^{t,x}$. The weak convergence to the second term is trivial since $h(X_T^{t,x})$ is independent of n. We then check the weak convergence to the last two terms. Let $\eta\in L_\rho^2(\Omega\otimes[t,T]\otimes\mathbb{R}^d;\mathbb{R}^1)$. Then noticing $\int_t^T\sup_n E[\int_s^T\int_{\mathbb{R}^d}|U_r^{t,x,n}|^2\rho^{-1}(x)dxdr]ds<\infty$ due to (3.7), by Lebesgue's dominated convergence theorem, we have

$$\begin{aligned}&|E[\int_t^T\int_{\mathbb{R}^d}\int_s^T(U_r^{t,x,n}-U_r^{t,x})dr\eta(s,x)\rho^{-1}(x)dxds]|\\ =&|E[\int_t^T\int_s^T\int_{\mathbb{R}^d}(U_r^{t,x,n}-U_r^{t,x})\eta(s,x)\rho^{-1}(x)dxdrds]|\\ \leq&\int_t^T|E[\int_s^T\int_{\mathbb{R}^d}(U_r^{t,x,n}-U_r^{t,x})\eta(s,x)\rho^{-1}(x)dxdr]|ds\longrightarrow 0,\quad \text{as } n\to\infty.\end{aligned}$$

On the other hand we know for fixed s and x, $\eta(s,x)\in L^2(\Omega)$. So there exists $\varphi(s,x,r)$ s.t. $\eta(s,x)=E[\eta(s,x)]+\int_t^T\langle\varphi(s,x,r),dW_r\rangle$. It is easy to see that for a.e. $s\in[t,T]$, $\varphi(s,\cdot,\cdot)\in$

$L^2(\Omega\otimes[t,T]\otimes\mathbb{R}^d;\mathbb{R}^1)$. Noticing that $\int_t^T\sup_n E[\int_s^T\int_{\mathbb{R}^d}|Z_r^{t,x,n}|^2\rho^{-1}(x)dxdr]ds<\infty$ due to (3.7) and using Lebesgue's dominated convergence theorem again, we obtain

$$\begin{aligned}
&|E[\int_t^T\int_{\mathbb{R}^d}\int_s^T\langle Z_r^{t,x,n}-Z_r^{t,x},dW_r\rangle\eta(s,x)\rho^{-1}(x)dxds]|\\
=&|\int_t^T\int_{\mathbb{R}^d}E[\int_s^T\langle Z_r^{t,x,n}-Z_r^{t,x},dW_r\rangle(E[\eta(s,x)]+\int_t^T\langle\varphi(s,x,r),dW_r\rangle)]\rho^{-1}(x)dxds|\\
=&|\int_t^T\int_{\mathbb{R}^d}E[\int_s^T\langle Z_r^{t,x,n}-Z_r^{t,x},\varphi(s,x,r)\rangle dr]\rho^{-1}(x)dxds|\\
\leq&\int_t^T|E[\int_s^T\int_{\mathbb{R}^d}\langle Z_r^{t,x,n}-Z_r^{t,x},\varphi(s,x,r)\rangle\rho^{-1}(x)dxdr]|ds\longrightarrow 0,\quad\text{as } n\to\infty.
\end{aligned}$$

Needless to say, if we can show BSDE (3.2) is indeed BSDE (2.2), then we can say $(Y_s^{t,x},Z_s^{t,x})$ is a solution of BSDE (2.2). The key is to prove that $U_s^{t,x}=f(s,X_s^{t,x},Y_s^{t,x},Z_s^{t,x})$ for a.a. $s\in[t,T]$, $x\in\mathbb{R}^d$ a.s. However, the weak convergence of Y^n, U^n and Z^n are not enough to this. The crucial point in this analysis is to establish the strong convergence of Y^n and Z^n, which will be done in next section.

4 The strong convergence and the identification of the limiting BSDEs

In this section, we will show that the combination of methods of weak convergence and strong convergence of a subsequence $(Y_s^{t,x,n},Z_s^{t,x,n})$ gives an effective way to prove that the limit $(Y_s^{t,x},Z_s^{t,x})$ satisfies BSDE (2.2). In contrast, the direct proof that BSDE (3.2) converges strongly to BSDE (2.2) by using the strongly convergent subsequence $(Y_s^{t,x,n},Z_s^{t,x,n})$ without the weak convergence argument will encounter some complications. This is due to that the dominated convergence theorem does not seem to apply immediately to BSDE (3.2). We start from an easy lemma.

Lemma 4.1 *Under the conditions of Theorem 2.3, if $u_n(t,x)$ is the weak solution of PDE (3.3), then $\sup_n\int_0^T\int_{\mathbb{R}^d}|u_n(s,x)|^{2p}\rho^{-1}(x)dxds<\infty$. Furthermore,*

$$\lim_{N\to\infty}\sup_n\int_0^T\int_{\mathbb{R}^d}|u_n(s,x)|^2I_{U_N{}^c}(x)\rho^{-1}(x)dxds=0,$$

where $U_N{}^c=\{x\in\mathbb{R}^d:\ |x|>N\}$.

Proof. The L_ρ^{2p} integrability of u_n follows directly from the equivalence of norm principle and Lemma 3.3. Let's prove the second part of this lemma. Since $\int_{\mathbb{R}^d}\rho^{-1}(x)dx<\infty$,

$$\begin{aligned}
&\lim_{N\to\infty}\sup_n\int_0^T\int_{\mathbb{R}^d}|u_n(s,x)|^2I_{U_N{}^c}(x)\rho^{-1}(x)dxds\\
\leq&\lim_{N\to\infty}\big(\sup_n\int_0^T\int_{\mathbb{R}^d}|u_n(s,x)|^{2p}\rho^{-1}(x)dxds\big)^{\frac{1}{p}}\big(\int_0^T\int_{\mathbb{R}^d}|I_{U_N{}^c}(x)|^{\frac{p}{p-1}}\rho^{-1}(x)dxds\big)^{\frac{p-1}{p}}\\
\leq&\lim_{N\to\infty}C_p\big(\int_{\mathbb{R}^d}I_{U_N{}^c}(x)\rho^{-1}(x)dx\big)^{\frac{p-1}{p}}=0.
\end{aligned}$$

◇

The following two theorems quoted in [21] will be used in this section.

Theorem 4.2 *(c.f. [21]) Let $X \subset\subset H \subset Y$ be Banach spaces, with X reflexive. Here $X \subset\subset H$ means X is compactly embedded in H. Suppose that u_n is a sequence that is uniformly bounded in $L^2([0,T];X)$, and du_n/dt is uniformly bounded in $L^p(0,T;Y)$, for some $p>1$. Then there is a subsequence that converges strongly in $L^2([0,T];H)$.*

Theorem 4.3 *(Rellich-Kondrachov Compactness Theorem c.f. [21]) Let B be a bounded C^1 domain in $\mathbb{R}^d$. Then $H^1(B)$ is compactly embedded in $L^2(B)$.*

Lemma 4.4 *Under the conditions of Theorem 2.3, if $(Y_s^{t,x,n}, Z_s^{t,x,n})$ is the solution of BSDEs (3.2) and $Y_s^{t,x}$ is the weak limit of $Y_s^{t,x,n}$ in $L^2_\rho(\Omega\otimes[t,T]\otimes\mathbb{R}^d;\mathbb{R}^1)$, then there is a subsequence of $Y_s^{t,x,n}$, still denoted by $Y_s^{t,x,n}$, converges strongly to $Y_s^{t,x}$ in $L^2(\Omega\otimes[t,T];L^2_\rho(\mathbb{R}^d;\mathbb{R}^1))$. Moreover, we have*

$$Y_s^{t,x} = Y_s^{s,X_s^{t,x}},\ Z_s^{t,x} = Z_s^{s,X_s^{t,x}} \quad \text{for any } s\in[t,T],\ \text{a.a. } x\in\mathbb{R}^d \text{ a.s.} \tag{4.1}$$

and $E[\int_t^T\int_{\mathbb{R}^d}|Y_s^{t,x}|^{2p}\rho^{-1}(x)dxds]<\infty$.

Proof. Let $u_n(s,x)=Y_s^{s,x,n}$. Then by Proposition 3.1, $u_n(s,X_s^{t,x,n})=Y_s^{t,x,n}$, $(\sigma^*\nabla u_n)(s,X_s^{t,x,n}) = Z_s^{t,x,n}$ for a.a. $s\in[t,T]$, $x\in\mathbb{R}^d$ a.s. We claim that $u_n(s,x)$ satisfies the following PDE in ${H_\rho^1}^*(\mathbb{R}^d;\mathbb{R}^1)$:

$$du_n(s,x)/ds = -\mathscr{L}u_n(s,x) - f_n\big(s,x,u_n(s,x),(\sigma^*\nabla u_n)(s,x)\big), \quad 0\le s\le T. \tag{4.2}$$

To prove this claim, first note that u_n are uniformly bounded in $L^2([0,T];H^1_\rho(\mathbb{R}^d;\mathbb{R}^1))$ by the equivalence of norm principle and the uniform ellipticity condition of σ:

$$\begin{aligned}
&\sup_n\int_0^T\int_{\mathbb{R}^d}(|u_n(s,x)|^2+|\nabla u_n(s,x)|^2)\rho^{-1}(x)dxds\\
&\le C_p\sup_n\int_0^T\int_{\mathbb{R}^d}(|u_n(s,x)|^2+|(\sigma^*\nabla u_n)(s,x)|^2)\rho^{-1}(x)dxds\\
&\le C_p\sup_n E[\int_0^T\int_{\mathbb{R}^d}(|Y_s^{0,x,n}|^2+|Z_s^{0,x,n}|^2)\rho^{-1}(x)dxds]<\infty.
\end{aligned} \tag{4.3}$$

Then we can deduce that du_n/ds are uniformly bounded in $L^2([0,T];{H_\rho^1}^*(\mathbb{R}^d;\mathbb{R}^1))$. For this, we need to prove that $\mathscr{L}u_n$ and $f_n\in L^2([0,T];{H_\rho^1}^*(\mathbb{R}^d;\mathbb{R}^1))$ are uniformly bounded respectively. First note that for $i=1,2,\cdots,d$,

$$|\frac{\partial\rho^{-1}(x)}{\partial x_i}| = |\frac{-qx_i}{(1+|x|)^{q+1}|x|}| \le \frac{q}{(1+|x|)^{q+1}} \le q\rho^{-1}(x).$$

Moreover, recalling the form of $\mathscr{L}$ and noticing the conditions on b and σ in (H.5), we can see that a_{ij} and b_i are uniformly bounded for all i, j. So for arbitrary $s\in[0,T]$, $\psi\in C_c^\infty(\mathbb{R}^d;\mathbb{R}^1)$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mathscr{L}u_n(s,x)\cdot\psi(x)\rho^{-1}(x)dx\\
&=\int_{\mathbb{R}^d}\big(-\frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial u_n(s,x)}{\partial x_i}\frac{\partial(a_{ij}\psi\rho^{-1})(x)}{\partial x_j}-\sum_{i=1}^{d}u_n(s,x)\frac{\partial(b_i\psi\rho^{-1})(x)}{\partial x_i}\big)dx\\
&\leq\int_{\mathbb{R}^d}(\sum_{i=1}^{d}|\frac{\partial u_n(s,x)}{\partial x_i}|+|u_n(s,x)|)(\sum_{i,j=1}^{d}|\frac{\partial(a_{ij}\psi\rho^{-1})(x)}{\partial x_j}|+\sum_{i=1}^{d}|\frac{\partial(b_i\psi\rho^{-1})(x)}{\partial x_i}|)dx\\
&\leq C_p\int_{\mathbb{R}^d}(\sum_{i=1}^{d}|\frac{\partial u_n(s,x)}{\partial x_i}|+|u_n(s,x)|)(\sum_{j=1}^{d}|\frac{\partial\psi(x)}{\partial x_j}|+|\psi(x)|)\rho^{-1}(x)dx\\
&\leq C_p\sqrt{\int_{\mathbb{R}^d}(\sum_{i=1}^{d}|\frac{\partial u_n(s,x)}{\partial x_i}|+|u_n(s,x)|)^2\rho^{-1}(x)dx}\sqrt{\int_{\mathbb{R}^d}(\sum_{j=1}^{d}|\frac{\partial\psi(x)}{\partial x_j}|+|\psi(x)|)^2\rho^{-1}(x)dx}\\
&\leq C_p\|u_n(s,x)\|_{H^1_\rho(\mathbb{R}^d;\mathbb{R}^1)}\|\psi\|_{H^1_\rho(\mathbb{R}^d;\mathbb{R}^1)}.
\end{aligned}$$

As $C_c^\infty(\mathbb{R}^d;\mathbb{R}^1)$ is dense in $H^1_\rho(\mathbb{R}^d;\mathbb{R}^1)$, therefore for arbitrary $s\in[0,T]$, it follows that $\|\mathscr{L}u_n(s,\cdot)\|_{H^{1*}_\rho(\mathbb{R}^d;\mathbb{R}^1)}\leq C_p\|u_n(s,\cdot)\|_{H^1_\rho(\mathbb{R}^d;\mathbb{R}^1)}$ and by (4.3), we have

$$\sup_n\|\mathscr{L}u_n\|^2_{L^2([0,T];H^{1*}_\rho(\mathbb{R}^d;\mathbb{R}^1))}\leq C_p\sup_n\int_0^T\int_{\mathbb{R}^d}(|u_n(s,x)|^2+|\nabla u_n(s,x)|^2)\rho^{-1}(x)dxds<\infty.$$

Also using Lemma 3.3 and the equivalence of norm principle again, we obtain

$$\begin{aligned}
&\int_0^T\|f_n(s,\cdot,u_n(s,\cdot),(\sigma^*\nabla u_n)(s,\cdot))\|^2_{L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)}ds\\
&\leq C_pE[\int_0^T\int_{\mathbb{R}^d}(|f_0(s,x)|^2+|Y_s^{0,x,n}|^{2p}+|Z_s^{0,x,n}|^2)\rho^{-1}(x)dxds]<\infty.
\end{aligned}$$

Hence $f_n\in L^2([0,T];L^{2*}_\rho(\mathbb{R}^d;\mathbb{R}^1))\subset L^2([0,T];H^{1*}_\rho(\mathbb{R}^d;\mathbb{R}^1))$ and

$$\begin{aligned}
&\sup_n\|f_n\|^2_{L^2([0,T];H^{1*}_\rho(\mathbb{R}^d;\mathbb{R}^1))}\\
&\leq C_p\sup_n\int_0^T\|f_n(s,\cdot,u_n(s,\cdot),(\sigma^*\nabla u_n)(s,\cdot))\|^2_{L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)}ds\\
&\leq C_p\sup_n E[\int_0^T\int_{\mathbb{R}^d}(|f_0(s,x)|^2+|Y_s^{0,x,n}|^{2p}+|Z_s^{0,x,n}|^2)\rho^{-1}(x)dxds]<\infty.
\end{aligned}$$

Therefore we conclude that du_n/ds are uniformly bounded in $L^2([0,T];H^{1*}_\rho(\mathbb{R}^d;\mathbb{R}^1))$.

Noticing Theorem 4.3 and applying Theorem 4.2 with $X=H^1_\rho(U_1;\mathbb{R}^1)$, $H=L^2_\rho(U_1;\mathbb{R}^1)$ and $Y=H^{1*}_\rho(U_1;\mathbb{R}^1)$, we are able to extract a subsequence of $u_n(s,x)$, denoted by $u_{1n}(s,x)$, which converges strongly in $L^2([0,T];L^2_\rho(U_1;\mathbb{R}^1))$. It is obvious that this $u_{1n}(s,x)$ satisfies the conditions in Theorem 4.2. Applying Theorem 4.2 again, we are able to extract a subsequence of $u_{1n}(s,x)$, denoted by $u_{2n}(s,x)$, that converges strongly in $L^2([0,T];L^2_\rho(U_2;\mathbb{R}^1))$. Actually we can do this procedure for all U_i, $i=1,2,\cdots$. Now we pick up the diagonal sequence $u_{ii}(s,x)$, $i=1,2,\cdots$ and still denote this sequence by u_n for convenience. It is easy to see that u_n converges strongly in all $L^2([0,T];L^2_\rho(U_i;\mathbb{R}^1))$, $i=1,2,\cdots$. For arbitrary $\varepsilon>0$, noticing Lemma 4.1, we can find $j(\varepsilon)$ large enough such that

$$\sup_n \int_0^T \int_{U_{j(\varepsilon)}{}^c} 2|u_n(s,x)|^2 \rho^{-1}(x)dxds < \frac{\varepsilon}{3}.$$

For this $j(\varepsilon)$, there exists $n^*(\varepsilon) > 0$ s.t. when $m, n \geq n^*(\varepsilon)$, we know

$$\|u_m - u_n\|^2_{L^2([0,T];L^2_\rho(U_{j(\varepsilon)};\mathbb{R}^1))} = \int_0^T \int_{U_{j(\varepsilon)}} |u_m(s,x) - u_n(s,x)|^2 \rho^{-1}(x)dxds < \frac{\varepsilon}{3}.$$

Therefore as $m, n \geq n^*(\varepsilon)$,

$$\begin{aligned}
&\|u_m - u_n\|^2_{L^2([0,T];L^2_\rho(\mathbb{R}^d;\mathbb{R}^1))} \\
&\leq \int_0^T \int_{U_{j(\varepsilon)}} |u_m(s,x) - u_n(s,x)|^2 \rho^{-1}(x)dxds + \int_0^T \int_{U_{j(\varepsilon)}{}^c} (2|u_m(s,x)|^2 + 2|u_n(s,x)|^2)\rho^{-1}(x)dxds \\
&< \varepsilon.
\end{aligned}$$

That is to say u_n converges strongly in $L^2([0,T];L^2_\rho(\mathbb{R}^d;\mathbb{R}^1))$. Now using the equivalence of norm principle, we know as m, $n \to \infty$,

$$\begin{aligned}
&\|Y_s^{t,x,m} - Y_s^{t,x,n}\|^2_{L^2(\Omega\otimes[t,T];L^2_\rho(\mathbb{R}^d;\mathbb{R}^1))} \\
&= E[\int_t^T \int_{\mathbb{R}^d} |u_m(s,X_s^{t,x}) - u_n(s,X_s^{t,x})|^2 \rho^{-1}(x)dxds] \\
&\leq C_p \int_t^T \int_{\mathbb{R}^d} |u_m(s,x) - u_n(s,x)|^2 \rho^{-1}(x)dxds \longrightarrow 0. \qquad (4.4)
\end{aligned}$$

So the claim that $Y_s^{t,x,n}$ converges strongly in $L^2(\Omega\otimes[t,T];L^2_\rho(\mathbb{R}^d;\mathbb{R}^1))$ follows. But we know that $Y_s^{t,x}$ is the weak limit of $Y_s^{t,x,n}$ in $L^2(\Omega\otimes[t,T];L^2_\rho(\mathbb{R}^d;\mathbb{R}^1))$, therefore $Y_s^{t,x,n}$ converges strongly to $Y_s^{t,x}$ in $L^2(\Omega\otimes[t,T];L^2_\rho(\mathbb{R}^d;\mathbb{R}^1))$.

To see (4.1), first notice that in BSDE (3.8), $h(\cdot) \in L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)$ and $U_\cdot^{t,\cdot} \in M^2([t,T];L^2_\rho(\mathbb{R}^d;\mathbb{R}^1))$ are given, so there exists $(Y_\cdot^{t,\cdot}, Z_\cdot^{t,\cdot}) \in M^2([t,T];L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)) \otimes M^2([t,T];L^2_\rho(\mathbb{R}^d;\mathbb{R}^d))$ satisfying the spatial integral form of BSDE (3.8). By Lemma 3.3 and Proposition 3.4 in [24], $(Y_s^{t,x}, Z_s^{t,x}) \in S^2([t,T];L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)) \otimes M^2([t,T];L^2_\rho(\mathbb{R}^d;\mathbb{R}^d))$ is the unique solution of BSDE (3.8) and $Y_s^{t,x} = Y_s^{s,X_s^{t,x}}$, $Z_s^{t,x} = Z_s^{s,X_s^{t,x}}$ for any $s \in [t,T]$, a.a. $x \in \mathbb{R}^d$ a.s. If we define $Y_s^{s,x} = u(s,x)$, then we can prove the strong limit of $u_n(s,x)$ in $L^2([0,T];L^2_\rho(\mathbb{R}^d;\mathbb{R}^1))$ is $u(s,x)$ and $Y_s^{t,x} = u(s,X_s^{t,x})$ for a.a. $s \in [t,T]$, $x \in \mathbb{R}^d$ a.s. To prove this, we only need to see that, by the equivalence of norm principle,

$$\begin{aligned}
&\lim_{n\to\infty} \int_0^T \int_{\mathbb{R}^d} |u_n(s,x) - u(s,x)|^2 \rho^{-1}(x)dxds \\
&\leq \lim_{n\to\infty} C_p E[\int_0^T \int_{\mathbb{R}^d} |u_n(s,X_s^{0,x}) - Y_s^{s,X_s^{0,x}}|^2 \rho^{-1}(x)dxds] \\
&= \lim_{n\to\infty} C_p E[\int_0^T \int_{\mathbb{R}^d} |Y_s^{0,x,n} - Y_s^{0,x}|^2 \rho^{-1}(x)dxds] = 0, \qquad (4.5)
\end{aligned}$$

and

$$E[\int_t^T \int_{\mathbb{R}^d} |Y_s^{t,x} - u(s,X_s^{t,x})|^2 \rho^{-1}(x)dxds]$$

$$
\begin{aligned}
&= E[\int_t^T \int_{\mathbb{R}^d} |Y_s^{s,X_s^{t,x}} - u(s,X_s^{t,x})|^2 \rho^{-1}(x)dxds] \\
&\leq C_p \int_t^T \int_{\mathbb{R}^d} |Y_s^{s,x} - u(s,x)|^2 \rho^{-1}(x)dxds = 0. \qquad (4.6)
\end{aligned}
$$

Moreover, we can prove that $E[\int_t^T \int_{\mathbb{R}^d} |Y_s^{t,x}|^{2p} \rho^{-1}(x)dxds] < \infty$. For this, by the equivalence of norm principle, we only need to prove that $\int_0^T \int_{\mathbb{R}^d} |u(s,x)|^{2p} \rho^{-1}(x)dxds < \infty$. To assert the claim, we first prove that we can find a subsequence of $\{u_n(s,x)\}_{n=1}^\infty$ still denoted by $\{u_n(s,x)\}_{n=1}^\infty$, s.t.

$$
u_n(s,x) \longrightarrow u(s,x) \text{ and } \sup_n |u_n(s,x)|^{2p} < \infty \text{ for a.e. } s \in [t,T],\ x \in \mathbb{R}^d. \qquad (4.7)
$$

For this, from (4.5), we know that $\int_0^T \int_{\mathbb{R}^d} |u_n(s,x) - u(s,x)|^2 \rho^{-1}(x)dxds \longrightarrow 0$. Therefore we may assume without losing any generality that $u_n(s,x) \longrightarrow u(s,x)$ for a.e. $s \in [0,T]$, $x \in \mathbb{R}^d$ and extract a subsequence of $\{u_n(s,x)\}_{n=1}^\infty$, still denoted by $\{u_n(s,x)\}_{n=1}^\infty$, s.t.

$$
\int_0^T \int_{\mathbb{R}^d} |u_{n+1}(s,x) - u_n(s,x)| \rho^{-1}(x)dxds \leq \frac{1}{2^n}.
$$

For any n,

$$
|u_n(s,x)| \leq |u_1(s,x)| + \sum_{i=1}^{n-1} |u_{i+1}(s,x) - u_i(s,x)| \leq |u_1(s,x)| + \sum_{i=1}^{\infty} |u_{i+1}(s,x) - u_i(s,x)|.
$$

Then by the triangle inequality of the norm, we have

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}^d} \sup_n |u_n(s,x)| \rho^{-1}(x)dxds \\
&\leq \int_0^T \int_{\mathbb{R}^d} |u_1(s,x)| \rho^{-1}(x)dxds + \sum_{i=1}^{\infty} \int_0^T \int_{\mathbb{R}^d} |u_{i+1}(s,x) - u_i(s,x)| \rho^{-1}(x)dxds \\
&\leq \int_0^T \int_{\mathbb{R}^d} |u_1(s,x)| \rho^{-1}(x)dxds + \sum_{i=1}^{\infty} \frac{1}{2^i} < \infty.
\end{aligned}
$$

Therefore, (4.7) follows from the above. By a similar argument as in Lemma 4.1, for this subsequence u_n, we can prove that for any $\delta > 0$,

$$
\lim_{N\to\infty} \sup_n \int_0^T \int_{\mathbb{R}^d} |u_n(s,x)|^{2p-\delta} I_{\{|u_n(s,x)|^{2p-\delta} > N\}}(s,x) \rho^{-1}(x)dxds = 0.
$$

That is to say that $|u_n(s,x)|^{2p}$ is uniformly integrable. Together with $u_n(s,x) \longrightarrow u(s,x)$ for a.e. $s \in [0,T]$, $x \in \mathbb{R}^d$, we have

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}^d} |u(s,x)|^{2p-\delta} \rho^{-1}(x)dxds = \lim_{n\to\infty} \int_0^T \int_{\mathbb{R}^d} |u_n(s,x)|^{2p-\delta} \rho^{-1}(x)dxds \\
&\leq \sup_n \int_0^T \int_{\mathbb{R}^d} |u_n(s,x)|^{2p-\delta} \rho^{-1}(x)dxds \leq C_p \big(\sup_n \int_0^T \int_{\mathbb{R}^d} |u_n(s,x)|^{2p} \rho^{-1}(x)dxds\big)^{\frac{2p-\delta}{2p}} \leq C_p,
\end{aligned}
$$

where the last $C_p < \infty$ is a constant independent of n and δ. Then using Fatou lemma to take the limit as $\delta \to 0$ in the above inequality, we can get $\int_0^T \int_{\mathbb{R}^d} |u(s,x)|^{2p} \rho^{-1}(x)dxds < \infty$. $\diamond$

Considering the strongly convergent subsequence $\{Y_\cdot^{t,\cdot,n}\}_{n=1}^\infty$ and using a standard argument to BSDE (3.2), we can prove that for arbitrary m, n

$$E[\int_{\mathbb{R}^d}|Y_s^{t,x,m}-Y_s^{t,x,n}|^2\rho^{-1}(x)dx]+\frac{1}{2}E[\int_s^T\int_{\mathbb{R}^d}|Z_r^{t,x,m}-Z_r^{t,x,n}|^2\rho^{-1}(x)dxdr]$$
$$\leq 2L^2E[\int_s^T\int_{\mathbb{R}^d}|Y_r^{t,x,m}-Y_r^{t,x,n}|^2\rho^{-1}(x)dxdr]+2\sqrt{E[\int_s^T\int_{\mathbb{R}^d}|Y_r^{t,x,m}-Y_r^{t,x,n}|^2\rho^{-1}(x)dxdr]}$$
$$\times C_p\sqrt{E[\int_s^T\int_{\mathbb{R}^d}(|f_0(r,x)|^2+|Y_r^{t,x,n}|^{2p}+|Z_r^{t,x,n}|^2)\rho^{-1}(x)dxdr]}.$$

So by Condition (H.2) and Lemma 3.3, we can conclude that the corresponding subsequence of $\{Z_\cdot^{t,\cdot,n}\}_{n=1}^\infty$ converges strongly as well. Certainly the strong convergence limit should be identified with the weak convergence limit $Z_\cdot^{t,\cdot}$, hence the following corollary follows without a surprise.

Corollary 4.5 *Let $(Y_\cdot^{t,\cdot}, Z_\cdot^{t,\cdot})$ be the solution to BSDE (2.2) and $(Y_\cdot^{t,\cdot,n}, Z_\cdot^{t,\cdot,n})$ be the subsequence of the solutions to BSDE (3.2), which $Y_\cdot^{t,\cdot,n}$ converges strongly to $Y_\cdot^{t,\cdot}$ in $L^2(\Omega\otimes[t,T];L_\rho^2(\mathbb{R}^d;\mathbb{R}^1))$, then $Z_\cdot^{t,\cdot,n}$ also converges strongly to $Z_\cdot^{t,\cdot}$ in $L^2(\Omega\otimes[t,T];L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$.*

From Lemma 4.4, we know that there is a subsequence of $Y_\cdot^{t,\cdot,n}$, still denoted by $Y_\cdot^{t,\cdot,n}$, converges strongly to $Y_\cdot^{t,\cdot}$ in $L^2(\Omega\otimes[t,T];L_\rho^2(\mathbb{R}^d;\mathbb{R}^1))$, i.e. $M^2([t,T];L_\rho^2(\mathbb{R}^d;\mathbb{R}^1))$. Indeed, doing Itô's formula to $\psi_M(Y_r^{t,x,n}-Y_r^{t,x})$ and $\mathrm{e}^{Kr}\varphi_{n,m}(\psi_M(Y_r^{t,x}))$, with Corollary 4.5 we can further prove that $Y_\cdot^{t,\cdot,n}$ converges to $Y_\cdot^{t,\cdot}$ in $S^2([t,T];L_\rho^2(\mathbb{R}^d;\mathbb{R}^1))$ and $Y_\cdot^{t,\cdot}\in S^{2p}([t,T];L_\rho^{2p}(\mathbb{R}^d;\mathbb{R}^1))$ respectively (To see similar calculations, one can refer to the argument in the proof of Lemma 3.3 in [24]).

Proposition 4.6 *For $(Y_\cdot^{t,\cdot}, Z_\cdot^{t,\cdot})$ and $(Y_\cdot^{t,\cdot,n}, Z_\cdot^{t,\cdot,n})$ given in Corollary 4.5, $Y_\cdot^{t,\cdot}$ is the limit of $Y_\cdot^{t,\cdot,n}$ in $S^2([t,T];L_\rho^2(\mathbb{R}^d;\mathbb{R}^1))$. Moreover, $Y_\cdot^{t,\cdot}\in S^{2p}([t,T];L_\rho^{2p}(\mathbb{R}^d;\mathbb{R}^1))$.*

Now we are ready to prove the identification of the limiting BSDEs.

Lemma 4.7 *The random field U, Y and Z have the following relation:*

$$U_s^{t,x}=f(s,X_s^{t,x},Y_s^{t,x},Z_s^{t,x})\ \text{ for a.a. } s\in[t,T],\ x\in\mathbb{R}^d \text{ a.s.} \tag{4.8}$$

Proof. Let $\mathcal{K}$ be a set in $\Omega\otimes[t,T]\otimes\mathbb{R}^d$ s.t. $\sup_n|Y_s^{t,x,n}|+\sup_n|Z_s^{t,x,n}|+|f_0(s,X_s^{t,x})|<K$. Similar to the proof of (4.7), we can find a subsequence of $\{(Y_s^{t,x,n},Z_s^{t,x,n})\}_{n=1}^\infty$, still denoted by $\{(Y_s^{t,x,n},Z_s^{t,x,n})\}_{n=1}^\infty$, satisfying $(Y_s^{t,x,n},Z_s^{t,x,n})\longrightarrow(Y_s^{t,x},Z_s^{t,x})$ and $\sup_n|Y_\cdot^{t,\cdot,n}|+\sup_n|Z_\cdot^{t,\cdot,n}|<\infty$ for a.e. $s\in[t,T]$, $x\in\mathbb{R}^d$ a.s. Then it turns out that as $K\to\infty$, $\mathcal{K}\uparrow\Omega\otimes[t,T]\otimes\mathbb{R}^d$. Moreover it is easy to see that along the subsequence,

$$E[\int_t^T\int_{\mathbb{R}^d}2(\sup_n|f_n(s,X_s^{t,x},Y_s^{t,x,n},Z_s^{t,x,n})|^2+|f(s,X_s^{t,x},Y_s^{t,x},Z_s^{t,x})|^2)I_{\mathcal{K}}(s,x)\rho^{-1}(x)dxds]$$
$$\leq 6C^2E[\int_t^T\int_{\mathbb{R}^d}(|f_0(s,X_s^{t,x})|^2+\sup_n|Y_s^{t,x,n}|^{2p}+\sup_n|Z_s^{t,x,n}|^2)I_{\mathcal{K}}(s,x)\rho^{-1}(x)dxds]$$
$$+6C^2E[\int_t^T\int_{\mathbb{R}^d}(|f_0(s,X_s^{t,x})|^2+|Y_s^{t,x}|^{2p}+|Z_s^{t,x}|^2)I_{\mathcal{K}}(s,x)\rho^{-1}(x)dxds]<\infty.$$

Thus, we can apply Lebesgue's dominated convergence theorem to the following calculation:

$$
\begin{aligned}
&\lim_{n\to\infty} E[\int_t^T\int_{\mathbb{R}^d} |f_n(s,X_s^{t,x},Y_s^{t,x,n},Z_s^{t,x,n})I_{\mathcal{K}}(s,x)-f(s,X_s^{t,x},Y_s^{t,x},Z_s^{t,x})I_{\mathcal{K}}(s,x)|^2\rho^{-1}(x)dxds]\\
&=E[\int_t^T\int_{\mathbb{R}^d}\lim_{n\to\infty}|f_n(s,X_s^{t,x},Y_s^{t,x,n},Z_s^{t,x,n})-f(s,X_s^{t,x},Y_s^{t,x},Z_s^{t,x})|^2I_{\mathcal{K}}(s,x)\rho^{-1}(x)dxds]\\
&\le 2E[\int_t^T\int_{\mathbb{R}^d}\lim_{n\to\infty}|f_n(s,X_s^{t,x},Y_s^{t,x,n},Z_s^{t,x,n})-f(s,X_s^{t,x},Y_s^{t,x,n},Z_s^{t,x,n})|^2I_{\mathcal{K}}(s,x)\rho^{-1}(x)dxds]\\
&\quad +2E[\int_t^T\int_{\mathbb{R}^d}\lim_{n\to\infty}|f(s,X_s^{t,x},Y_s^{t,x,n},Z_s^{t,x,n})-f(s,X_s^{t,x},Y_s^{t,x},Z_s^{t,x})|^2I_{\mathcal{K}}(s,x)\rho^{-1}(x)dxds].
\end{aligned}
\tag{4.9}
$$

Since $Y_s^{t,x,n}\longrightarrow Y_s^{t,x}$ for a.e. $s\in[t,T]$, $x\in\mathbb{R}^d$ a.s., there exists a $N(s,x,\omega)$ s.t. when $n\ge N(s,x,\omega)$, $|Y_s^{t,x,n}|\le|Y_s^{t,x}|+1$. So taking $n\ge\max\{N(s,x,\omega),\ |Y_s^{t,x}|+1\}$, we have $f_n(s,X_s^{t,x},Y_s^{t,x,n},Z_s^{t,x,n}) = f(s,X_s^{t,x},\frac{\inf(n,|Y_s^{t,x,n}|)}{|Y_s^{t,x,n}|}Y_s^{t,x,n},Z_s^{t,x,n}) = f(s,X_s^{t,x},Y_s^{t,x,n},Z_s^{t,x,n})$. That is to say $\lim_{n\to\infty}|f_n(s,X_s^{t,x},Y_s^{t,x,n},Z_s^{t,x,n})-f(s,X_s^{t,x},Y_s^{t,x,n},Z_s^{t,x,n})|^2=0$ for a.e. $s\in[t,T]$, $x\in\mathbb{R}^d$ a.s. On the other hand, $\lim_{n\to\infty}|f(s,X_s^{t,x},Y_s^{t,x,n},Z_s^{t,x,n})-f(s,X_s^{t,x},Y_s^{t,x},Z_s^{t,x})|^2=0$ for a.e. $s\in[t,T]$, $x\in\mathbb{R}^d$ a.s. is obvious due to the continuity of $(y,z)\to f(s,x,y,z)$.

Therefore by (4.9), $f_n(s,X_s^{t,x},Y_s^{t,x,n},Z_s^{t,x,n})I_{\mathcal{K}}(s,x)=U_s^{t,x,n}I_{\mathcal{K}}(s,x)$ converges strongly to $f(s,X_s^{t,x},Y_s^{t,x},Z_s^{t,x})I_{\mathcal{K}}(s,x)$ in $L_\rho^2(\Omega\otimes[t,T]\otimes\mathbb{R}^d;\mathbb{R}^1)$, but $U_s^{t,x,n}I_{\mathcal{K}}(s,x)$ converges weakly to $U_s^{t,x}I_{\mathcal{K}}(s,x)$ in $L_\rho^2(\Omega\otimes[t,T]\otimes\mathbb{R}^d;\mathbb{R}^1)$, so $f(s,X_s^{t,x},Y_s^{t,x},Z_s^{t,x})I_{\mathcal{K}}(s,x)=U_s^{t,x}I_{\mathcal{K}}(s,x)$ for a.e. $r\in[t,T]$, $x\in\mathbb{R}^d$ a.s. The lemma follows when $K\to\infty$. ⋄

Proof of Theorem 2.3. With Lemma 4.7 and Proposition 4.6, the existence of solutions to BSDE (2.2) is easy to see. Now we prove the uniqueness. If there is another solution $(\tilde Y_s^{t,x},\tilde Z_s^{t,x})$ to BSDE (2.2), then for a.e. $x\in\mathbb{R}^d$, $(Y_s^{t,x}-\tilde Y_s^{t,x},Z_s^{t,x}-\tilde Z_s^{t,x})$ satisfies

$$
Y_s^{t,x}-\tilde Y_s^{t,x}=\int_s^T\big(f(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})-f(r,X_r^{t,x},\tilde Y_r^{t,x},\tilde Z_r^{t,x})\big)dr-\int_s^T\langle Z_r^{t,x}-\tilde Z_r^{t,x},dW_r\rangle.
$$

Applying Itô's formula to $|Y_s^{t,x}-\tilde Y_s^{t,x}|^2$, by the stochastic Fubini theorem and Conditions (H.3)* and (H.4), we have

$$
\begin{aligned}
&E[\int_{\mathbb{R}^d}|Y_s^{t,x}-\tilde Y_s^{t,x}|^2\rho^{-1}(x)dx]+E[\int_s^T\int_{\mathbb{R}^d}|Z_r^{t,x}-\tilde Z_r^{t,x}|^2\rho^{-1}(x)dxdr]\\
&\le 2L^2E[\int_s^T\int_{\mathbb{R}^d}|Y_r^{t,x}-\tilde Y_r^{t,x}|^2\rho^{-1}(x)dxdr]+\frac12E[\int_s^T\int_{\mathbb{R}^d}|Z_r^{t,x}-\tilde Z_r^{t,x}|^2\rho^{-1}(x)dxdr].
\end{aligned}
$$

By Gronwall's inequality, the uniqueness of the solution to BSDE (2.2) follows immediately. ⋄

5 The PDEs

Now we make use of the results for BSDE (2.2) to give the probabilistic representation to PDEs with p-growth coefficients. Actually the solution of BSDE in the ρ-weighted L^2 space gives the unique weak solution of its corresponding PDE (2.4).

Proof of Theorem 2.4. Using Corollary 4.5, we first prove the relationship between (Y,Z) and u. Since (4.6), we only need to prove that $(\sigma^*\nabla u)(s,X_s^{t,x}) = Z_s^{t,x}$ for a.a. $s\in[t,T]$, $x\in\mathbb{R}^d$ a.s. This can be deduced from (4.1) and the strong convergence of $Z_\cdot^{t,\cdot,n}$ to $Z_\cdot^{t,\cdot}$ in $L^2(\Omega\otimes[t,T];L^2_\rho(\mathbb{R}^d;\mathbb{R}^1))$ by the similar argument as in Proposition 4.2 in [24].

We then prove that $u(t,x)$ is the unique weak solution of PDE (2.4). We still start from PDE (3.3). Let $u^n(s,x)$ be the weak solution of PDE (3.3). Then by the definition for the weak solution of PDE, we know $(u_n,\sigma^*\nabla u_n)\in L^2([0,T];L^2_\rho(\mathbb{R}^d;\mathbb{R}^1))\bigotimes L^2([0,T];L^2_\rho(\mathbb{R}^d;\mathbb{R}^d))$ and for an arbitrary $\varphi\in C_c^\infty(\mathbb{R}^d;\mathbb{R}^1)$,

$$\begin{aligned}&\int_{\mathbb{R}^d}u_n(t,x)\varphi(x)dx-\int_{\mathbb{R}^d}u_n(T,x)\varphi(x)dx-\frac{1}{2}\int_t^T\int_{\mathbb{R}^d}\big((\sigma^*\nabla u_n)(s,x)\big)^*(\sigma^*\nabla\varphi)(x)dxds\\&-\int_t^T\int_{\mathbb{R}^d}u_n(s,x)div\big((b-\tilde{A})\varphi\big)(x)dxds\\=&\int_t^T\int_{\mathbb{R}^d}f_n\big(s,x,u_n(s,x),(\sigma^*\nabla u_n)(s,x)\big)\varphi(x)dxds.\end{aligned}\tag{5.1}$$

We can prove along a subsequence that each term of (5.1) converges to the corresponding term of (2.5). By (4.5), we know that u_n converges strongly to u in $L^2_\rho([0,T]\otimes\mathbb{R}^d;\mathbb{R}^1)$, thus u_n also converges weakly. Moreover, $\sup_{x\in\mathbb{R}^d}(|div\big((b-\tilde{A})\varphi\big)(x)|)<\infty$ and ρ is a continuous functional in $\mathbb{R}^d$, so it is obvious that

$$\lim_{n\to\infty}\int_t^T\int_{\mathbb{R}^d}u_n(s,x)div\big((b-\tilde{A})\varphi\big)(x)dxds=\int_t^T\int_{\mathbb{R}^d}u(s,x)div\big((b-\tilde{A})\varphi\big)(x)dxds.$$

Also it is easy to see that

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{2}\int_t^T\int_{\mathbb{R}^d}\big((\sigma^*\nabla u_n)(s,x)\big)^*(\sigma^*\nabla\varphi)(x)dxds\\=&\lim_{n\to\infty}-\frac{1}{2}\int_t^T\int_{\mathbb{R}^d}u_n(s,x)div(\sigma\sigma^*\nabla\varphi)(x)\rho(x)\rho^{-1}(x)dxds\\=&-\frac{1}{2}\int_t^T\int_{\mathbb{R}^d}u(s,x)div(\sigma\sigma^*\nabla\varphi\sigma)(x)\rho(x)\rho^{-1}(x)dxds\\=&\frac{1}{2}\int_t^T\int_{\mathbb{R}^d}\big((\sigma^*\nabla u)(s,x)\big)^*(\sigma^*\nabla\varphi)(x)dxds.\end{aligned}$$

Also we have proved that $f_n(s,X_s^{t,x},Y_s^{t,x,n},Z_s^{t,x,n})$ converges weakly to $f(s,X_s^{t,x},Y_s^{t,x},Z_s^{t,x})$ in $L^2_\rho(\Omega\otimes[t,T]\otimes\mathbb{R}^d;\mathbb{R}^1)$. In fact we can follow the same procedure as in the proof of Lemma 4.7 to prove $f_n\big(s,x,u_n(s,x),(\sigma^*\nabla u_n)(s,x)\big)$ converges weakly to $f\big(s,x,u(s,x),(\sigma^*\nabla u)(s,x)\big)$ in $L^2_\rho([t,T]\otimes\mathbb{R}^d;\mathbb{R}^1)$. So we have

$$\begin{aligned}&\lim_{n\to\infty}\int_t^T\int_{\mathbb{R}^d}f_n\big(s,x,u_n(s,x),(\sigma^*\nabla u_n)(s,x)\big)\varphi(x)dxds\\=&\int_t^T\int_{\mathbb{R}^d}f\big(s,x,u(s,x),(\sigma^*\nabla u)(s,x)\big)\varphi(x)dxds.\end{aligned}$$

For any $t \in [0,T]$, $\lim_{n\to\infty}\int_{\mathbb{R}^d} u_n(t,x)\varphi(x)dx = \int_{\mathbb{R}^d} u(t,x)\varphi(x)dx$ can be proved as follows using Proposition 4.6:

$$\begin{aligned}\lim_{n\to\infty} \Big|\int_{\mathbb{R}^d}(u_n(t,x)-u(t,x))\varphi(x)dx\Big|^2 &\le \lim_{n\to\infty} C_pE[\int_{\mathbb{R}^d}|u_n(t,X_t^{0,x})-u(t,X_t^{0,x})|^2\rho^{-1}(x)dx]\\ &\le \lim_{n\to\infty} C_pE[\sup_{0\le t\le T}\int_{\mathbb{R}^d}|Y_t^{0,x,n}-Y_t^{0,x}|^2\rho^{-1}(x)dx]=0.\end{aligned}$$

Therefore we can prove (2.5) is satisfied for all $t\in[0,T]$. That is to say $u(t,x)$ is a weak solution of PDE (2.4).

The uniqueness of PDE (2.4) can be derived from the uniqueness of BSDE (2.2). Let u be a solution of PDE (2.4). Define $F(s,x)=f\big(s,x,u(s,x),(\sigma^*\nabla u)(s,x)\big)$. Since u is the solution, so $\int_0^T\int_{\mathbb{R}^d}\big(|u(s,x)|^{2p}+|(\sigma^*\nabla u)(s,x)|^2\big)\rho^{-1}(x)dxds<\infty$ and

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^d}|F(s,x)|^2\rho^{-1}(x)dxds\\ &\le C_p\int_0^T\int_{\mathbb{R}^d}\big(|f_0(s,x)|^2+|u(s,x)|^{2p}+|(\sigma^*\nabla u)(s,x)|^2\big)\rho^{-1}(x)dxds<\infty. \qquad (5.2)\end{aligned}$$

If we define $Y_s^{t,x}=u(s,X_s^{t,x})$ and $Z_s^{t,x}=(\sigma^*\nabla u)(s,X_s^{t,x})$, then by Lemma 3.2,

$$\begin{aligned}&E[\int_t^T\int_{\mathbb{R}^d}(|Y_s^{t,x}|^{2p}+|Z_s^{t,x}|^2)\rho^{-1}(x)dxds]\\ &\le C_p\int_t^T\int_{\mathbb{R}^d}|u(s,x)|^{2p}+|(\sigma^*\nabla u)(s,x)|^2\rho^{-1}(x)dxds<\infty.\end{aligned}$$

Using some ideas of Theorem 2.1 in [1], similar to the argument as in Section 4 in [24], we have for $t\le s\le T$, $(Y_\cdot^{t,\cdot},Z_\cdot^{t,\cdot})\in M^{2p}([t,T];L_\rho^{2p}(\mathbb{R}^d;\mathbb{R}^1))\bigotimes M^2([t,T];L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$ solves the following BSDE:

$$Y_s^{t,x}=h(X_T^{t,x})+\int_s^T F(r,X_r^{t,x})dr-\int_s^T\langle Z_r^{t,x},dW_r\rangle. \qquad (5.3)$$

Multiply $\varphi\in C_c^0(\mathbb{R}^d;\mathbb{R}^1)$ on both sides and then take the integration over $\mathbb{R}^d$. Noting the definition of $F(s,x)$, $Y_s^{t,x}$ and $Z_s^{t,x}$, we have that $(Y_s^{t,x},Z_s^{t,x})$ satisfies the spatial integration form of BSDE (2.2). Similar to Proposition 4.6, we can deduce that $Y_\cdot^{t,\cdot}\in S^{2p}([t,T];L_\rho^{2p}(\mathbb{R}^d;\mathbb{R}^1))$ and therefore $(Y_s^{t,x},Z_s^{t,x})$ is a solution of BSDE (2.2). If there is another solution $\hat{u}$ to PDE (2.4), then by the same procedure, we can find another solution $(\hat{Y}_s^{t,x},\hat{Z}_s^{t,x})$ to BSDE (2.2), where

$$\hat{Y}_s^{t,x}=\hat{u}(s,X_s^{t,x}) \text{ and } \hat{Z}_s^{t,x}=(\sigma^*\nabla\hat{u})(s,X_s^{t,x}).$$

By Theorem 2.3, the solution of BSDE (2.2) is unique. Therefore

$$Y_s^{t,x}=\hat{Y}_s^{t,x} \text{ for a.a. } s\in[t,T],\ x\in\mathbb{R}^d \text{ a.s.}$$

In particular, when $t=0$,

$$Y_s^{0,x}=\hat{Y}_s^{0,x} \text{ for a.a. } s\in[0,T],\ x\in\mathbb{R}^d \text{ a.s.}$$

By Lemma 3.2 again,

$$\int_0^T \int_{\mathbb{R}^d} |u(s,x)-\hat{u}(s,x)|^2\rho^{-1}(x)dxds \leq C_p E[\int_0^T \int_{\mathbb{R}^d} |Y_s^{0,x}-\hat{Y}_s^{0,x}|^2)\rho^{-1}(x)dxds] = 0.$$

So $u(s,x)=\hat{u}(s,x)$ for a.a. $s\in[0,T]$, $x\in\mathbb{R}^d$ a.s. The uniqueness is proved. ⋄

Acknowledgements. We would like to acknowledge useful conversations with K.D. Elworthy, C.R. Feng, J. Lorinczi, K.N. Lu, S.G. Peng and S.J. Tang. QZ would like to thank the Department of Mathematical Sciences of Loughborough University for appointing him as a Research Associate. He wishes to acknowledge their financial support to the project through the appointment and partial financial support of the National Basic Research Program of China (973 Program) with Grant No.2007CB814904. HZ would like to thank K.N. Lu for inviting him to visit Brigham-Young University under their programme of Special Year on Stochastic Dynamics and S.J. Tang for inviting him to visit Laboratory of Mathematics for Nonlinear Science, Fudan University.

References

1. V. Bally, A. Matoussi, *Weak solutions for SPDEs and backward doubly stochastic differential equations.* J. Theor. Probab., Vol.14 (2001), 125-164.
2. G. Barles, E. Lesigne, *SDE, BSDE and PDE.* In: Backward stochastic differential equations. Pitman Res. Notes Math., Ser.364, Longman, Harlow, (1997), 47-80.
3. Ph. Briand, Y. Hu, *BSDE with quadsatic growth and unbounded terminal value.* Probab. Theory Rel., Vol.136 (2006), 604-618.
4. R. C. Dalang, C. Mueller and R. Tribe, *A Feynman-Kac-type formula for the deterministic and stochastic wave equations and other P.D.E.'s.* Trans. AMS, Vol.360 (2008), 4681-4703.
5. R. W. R. Darling, *Constructing Gamma-martingales with prescribed limits, using backward SDE.* Ann. Probab., Vol.23 (1995), 1234-1261.
6. M. D. Donsker, S. R. S. Varadhan, *Asymptotic evaluation of certain Markov process expectations for large time, I and II.* Comm. Pure. Appl. Math., Vol.28 (1975), 1-47 and 279-301.
7. K. D. Elworthy, A. Truman, H. Z. Zhao, J. G. Gaines, *Approximate travelling waves for generalized KPP equations and classical mechanics.* Proc. R. Soc. London, Sect.A, Vol.446 (1994), 529-554.
8. R. P. Feynman, *Space-time approach to non-relativistic quantum mechanics.* Rev. Mod. Phys., Vol.20 (1948), 367-387.
9. M. I. Freidlin, *Functional integration and partial differential equations.* Annals of Mathematics Studies, Vol.109, Princeton University Press, Princeton, New Jersey (1985).
10. K. D. Elworthy, *Geometric aspects of diffusions on manifolds.* Ecole d'Eté de Probabilités de Saint Flour, XVII. Lecture Notes in Math. 1362 276-425. Springer, Berlin. (1987).
11. N. El-Karoui, S. Peng, M. C. Quenez, *Backward stochastic differential equations in finance.* Math. Finance, Vol.7 (1997), 1-72.
12. M. Kac, *On distributions of certain Wiener functionals.* Trans. AMS, Vol.65 (1949), 1-13.
13. M. Kobylanski, *Backward stochastic differential equations and partial differential equations with quadsatic growth.* Ann. Probab., Vol.28 (2000), 558-602.
14. H. Kunita, *Stochastic flow acting on Schwartz distributions.* J. Theor. Probab., Vol.7 (1994), 247-278.
15. J. P. Lepeltier, J. San Martin, *Backward stochastic differential equations with continuous coefficient.* Stat. Probab. Lett., Vol.32 (1997), 425-430.
16. P. Malliavin, D. W. Stroock, *Short time behaviour of the heat kernel and it logarithmic derivatives.* J. Differ. Geom., Vol.44 (1996), 550-570.

17. A. Matoussi and M. Xu, *Sobolev solution for semi-linear PDE with obstacle under monotonicity condition.* Electron. J. Probab., Vol.13 (2008), 1035-1067.
18. E. Pardoux, *BSDE's weak convergence and homogenization of semilinear PDE's.* In F. Clarke and R. Stern eds. Nonlin. Analy., Diff. Equa. and Control, Kluwer Acad. Publi. Dordrecht (1999), 503-549.
19. E. Pardoux, S. Peng, *Adapted solution of a backward stochastic differential equation.* Syst. Control Lett., Vol.14 (1990), 55-61.
20. E. Pardoux, S. Peng, *Backward stochastic differential equations and quasilinear parabolic partial differential equations.* Stochastic Partial Differential Equations, B. L. Rozuvskii, R. B. Sowers eds., Lect. Notes Control Inf. Sci., Berlin Heidelberg New York: Springer, Vol.176 (1992), 200-217.
21. J. C. Robinson, *Infinite-dimensional dynamical systems: an introduction to dissipative parabolic PDEs and the theory of global attractors.* Cambridge University Press (2001).
22. B. Simon, *Functional integration and quantum physics.* AMS Chelsea Publishing, 2nd Edition, (2005).
23. A. D. Wentzell, M. I. Freidlin, *On small random perturbations of dynamical system.* Russ. Math. Surv., Vol.25 (1970), 1-55.
24. Q. Zhang, H. Z. Zhao, *Stationary solutions of SPDEs and infinite horizon BDSDEs.* J. Funct. Anal., Vol.252 (2007), 171-219.
25. Q. Zhang, H. Z. Zhao, *Stationary solutions of SPDEs and infinite horizon BDSDEs with non-Lipschitz coefficients.* J. Differ. Equations, Vol.248 (2010), 953-991.