Pathwise Random Periodic Solutions of Stochastic Differential Equations

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Summary. In this paper, we study the existence of random periodic solutions for semilinear stochastic differential equations. We identify them as the solutions of coupled forward-backward infinite horizon stochastic integral equations in general cases. We then use the argument of the relative compactness of Wiener-Sobolev spaces in $C^0([0,T],L^2(\Omega))$ and generalized Schauder's fixed point theorem to prove the existence of a solution of the coupled stochastic forward-backward infinite horizon integral equations. The condition on F is then further weakened by applying coupling method of forward and backward Gronwall inequalities. The results are also valid for stationary solutions as a special case when the period τ can take an arbitrary number.

Keywords: random periodic solution, random dynamical system, semilinear stochastic differential equation, coupling method, relative compactness, Malliavin derivative, coupled forward-backward infinite horizon stochastic integral equations.

1 Introduction

Random dynamical systems arise in modeling many phenomena in physics, biology, climatology, economics, etc., when uncertainties or random influences, called noises, are taken into account. The need for studying random dynamical systems was presented by Ulam and von Neumann [26] in 1945. It has been pushed since 1980s by establishing random dynamical systems generated by random maps, stochastic ordinary differential equations and stochastic partial differential equations, we refer the reader to [1],[9],[10],[14],[17] and the references therein. Periodic solutions have been a central concept in the theory of the deterministic dynamical system for over a century starting from Poincaré's seminal work [21]. They have been studied for many important problems arising in numerous physical problems e.g. van der Pol equations ([27]), Liénard equations [13]. Now after over a century, this topic is still one of the most interesting nonlinear phenomena to study in the theory of the deterministic dynamical system. Periodic behaviour arises naturally in many real world problems e.g. in biological, enviromental and economic systems. But these problems often subject to random perturbations or under the influence of noises. Needless to say, for random dynamical systems, to study the pathwise random periodic solutions is of great interests and challenging. Zhao and Zheng [29] started to study the problem and gave a definition of the pathwise random periodic solutions for C^1-cocycles. It is well-known that in the deterministic case, the most powerful method to prove the existence of the periodic solution is to study the fixed point of the Poincaré map. However, for random dynamical systems, it is very difficult, if not impossible, to define a useful Poincaré map and to find its fixed point as the trajectory does not return to the same set with certainty. In this paper, we will study the τ-periodic solutions of τ-periodic stochastic differential equations in R^d:

$$du(t) = -Au(t)\,dt + F(t,u(t))\,dt + B_0(t)dW(t), \qquad t \geq s, \tag{1.1}$$

$$u(s) = x \in R^d.$$

Denote $\Delta := \{(t,s) \in R^2, s \leq t\}$. This equation generates a semi-flow $u : \Delta \times R^d \times \Omega \to R^d$ when the solution exists uniquely. Assume F and B_0 satisfy:

Condition (P) *There exists a constant $\tau > 0$ such that for any $t \in R$, $u \in R^d$*

$$F(t,u) = F(t+\tau, u), \ \ B_0(t) = B_0(t+\tau).$$

First, we give the definition of the random periodic solution

Definition 1.1 *A random periodic solution of period τ of a semi-flow $u : \Delta \times R^d \times \Omega \to R^d$ is an $\mathcal{F}$-measurable map $\varphi : (-\infty, \infty) \times \Omega \to R^d$ such that*

$$u(t+\tau, t, \varphi(t,\omega), \omega) = \varphi(t+\tau, \omega) = \varphi(t, \theta_\tau \omega), \tag{1.2}$$

for any $t \in R$ and $\omega \in \Omega$.

Instead of following the traditional geometric method of establishing the Poincaré mapping, for the stochastic semi-flow, we will give a new analysis method of coupled infinite horizon forward-backward integral equations. We will prove that the solution of the coupled forward-backward infinite horizon integral equation gives a random periodic solution of period τ and vise versa if the random periodic solution is tempered. Under certain conditions, we can solve this coupled forward-backward infinite horizon integral equations. For this, we use generalized Schauder's fixed point theorem and relative compactness argument in Wiener-Sobolev spaces of Malliavin derivatives. The stationary solution is also obtained as a special case when the period τ can take an arbitrary number. This is the case when the operators $F(t,u)$ and $B_0(t)$ do not depend on time t. In deterministic and random dynamical systems, to find the existence of stationary solutions and random periodic solutions, and to construct local stable and unstable manifolds near a hyperbolic stationary point is a fundamental problem ([1],[7],[8],[11],[12],[17],[23]). The stationary solution for the deterministic autonomous parabolic differential equations actually is a solution of the corresponding elliptic equation. This statement is not true for non-autonomous parabolic partial differential equations, even for the deterministic case with nonlinear terms periodic in time. For stochastic differential equations or stochastic partial differential equations with autonomous or time periodic nonlinear terms, to find a stationary solution or a random periodic solution is a more difficult and subtle problem. In fact, in literature, researchers usually assume there is an invariant set or a stationary solution or a fixed point, then prove invariant manifolds and stability results at a point of the invariant set ([1],[7],[8],[11],[17],[23]). So to know what the invariant set is and whether or not the invariant set is a stationary solution or a random periodic solution or has more complicated topology is a basic problem. In fact, for the existence of stationary solutions, results are only known in very few cases ([3],[6],[17],[24],[25],[28]). Even for the stationary solution case, researchers can only construct stable stationary solution using the convergence of the pullback of the solution or infinite horizon backward stochastic differential equations (e.g. [3], [15],[16],[28]). Our result actually gives a general method to establish bistable stationary solutions and random periodic solutions. For the periodic stochastic evolution case, as far as we know, this is the first paper investigating the random periodic solution, even it is a very natural problem. Since Theorem 2.1 is valid in very general situations, we believe the coupled infinite horizon forward-backward stochastic integral equations (2.2) should be useful in investigating random periodic solutions of many kinds of stochastic differential equations and stochastic partial differential equations.

2 Coupled Forward-Backward Infinite Horizon Stochastic Integral Equations and Random Periodic Solutions

We consider the semilinear stochastic differential equation (1.1). Denote the solution by $u(t,s,x,\omega)$. Let A be an $d\times d$ matrix, we can also regard it as a linear operator in $\mathcal{L}(R^d)$. Throughout this paper, we suppose that $T_t=e^{-At}$ is a hyperbolic linear flow. So R^d has a direct sum decomposition (c.f. [22]):

$$R^d=E^s\oplus E^u,$$

where

$$E^s=span\{v: v \text{ is a generalized eigenvector for an eigenvalue } \lambda \text{ with } Re(\lambda)<0\},$$

$$E^u=span\{v: v \text{ is a generalized eigenvector for an eigenvalue } \lambda \text{ with } Re(\lambda)>0\}.$$

Denote μ_m an eigenvalue of A with the largest negative real part, and μ_{m+1} with the smallest positive real part. We also define the projections onto each subspace by

$$P^+: R^d\to E^u,\ P^-: R^d\to E^s.$$

Let $W(t), t\in R$ be an M-dimensional Brownian motion and the filtered Wiener space is $(\Omega,\mathcal{F},(\mathcal{F}^t)_{t\in R},P)$. Here $\mathcal{F}_s^t:=\sigma(W_u-W_v, s\le v\le u\le t)$ and $\mathcal{F}^t:=\vee_{s\le t}\mathcal{F}_s^t$. Suppose $B_0(s)$ is an $d\times M$ matrix and is globally bounded $\sup_{-\infty<s<\infty}||B_0(s)||<\infty$. The solution of the initial value problem (1.1) is given by the following variation of constant formula:

$$u(t,s,x,\omega)=T_{t-s}x+\int_s^t T_{t-r}F(r,u(r,s,x,\omega))dr+\int_s^t T_{t-r}B_0(r)dW(r). \tag{2.1}$$

We consider a solution of the following coupled forward-backward infinite horizon stochastic integral equation, which is a $\mathcal{B}(R)\otimes\mathcal{F}$-measurable map $Y:(-\infty,\infty)\times\Omega\to R^d$ satisfying

$$\begin{aligned}Y(t,\omega)=&\int_{-\infty}^t T_{t-s}P^+F(s,Y(s,\omega))ds-\int_t^\infty T_{t-s}P^-F(s,Y(s,\omega))ds\\&+(\omega)\left[\int_{-\infty}^t T_{t-s}P^+B_0(s)\,dW(s)\right]-(\omega)\left[\int_t^\infty T_{t-s}P^-B_0(s)\,dW(s)\right]\end{aligned} \tag{2.2}$$

for all $\omega\in\Omega$, $t\in(-\infty,\infty)$. We will give the following general theorem which identifies the solution of the equation (2.2) and a random periodic solution of stochastic differential equation (1.1). First of all, we recall the definition of a tempered random variable (Definition 4.1.1 in [1]):

Definition 2.1 *A random variable $X:\Omega\to R^d$ is called tempered with respect to the dynamical system θ if*

$$\lim_{r\to\pm\infty}\frac{1}{|r|}\log|X(\theta_r\omega)|=0.$$

The random variable is called tempered from above (below) if in the above limit, the function log *is replaced by* $\log^+$ *($\log^-$), the positive (negative) part of the function* log.

Theorem 2.1 *Assume Condition (P). If Cauchy problem (1.1) has a unique solution $u(t,s,x,\omega)$ and the coupled forward-backward infinite horizon stochastic integral equation (2.2) has one solution*

$Y:(-\infty,+\infty)\times\Omega\to R^d$ such that $Y(t+\tau,\omega)=Y(t,\theta_\tau\omega)$ for any $t\in R$ a.s., then Y is a random periodic solution of equation (1.1) i.e.

$$u(t+\tau,t,Y(t,\omega),\omega)=Y(t+\tau,\omega)=Y(t,\theta_\tau\omega)\ \text{ for any }\ t\in R\quad a.s. \tag{2.3}$$

Conversely, if equation (1.1) has a random periodic solution $Y:(-\infty,+\infty)\times\Omega\to R^d$ of period τ which is tempered from above for each t, then Y is a solution of the coupled forward-backward infinite horizon stochastic integral equation (2.2).

Proof: If equation (2.2) has a solution such that $Y(t+\tau,\omega)=Y(t,\theta_\tau\omega)$ for any $t\in R$, then we have

$$\begin{aligned}
&Y(t+\tau,\omega)\\
&=Y(t,\theta_\tau\omega)\\
&=\int_{-\infty}^{t}T_{t-s}P^+F(s,Y(s,\theta_\tau(\omega)))ds-\int_t^\infty T_{t-s}P^-F(s,Y(s,\theta_\tau(\omega)))ds\\
&\quad+(\theta_\tau\omega)\int_{-\infty}^{t}T_{t-s}P^+B_0(s)dW(s)-(\theta_\tau\omega)\int_t^\infty T_{t-s}P^-B_0(s)dW(s)\\
&=\int_{-\infty}^{t}T_{t-s}P^+F(s+\tau,Y(s,\theta_\tau(\omega)))ds-\int_t^\infty T_{t-s}P^-F(s+\tau,Y(s,\theta_\tau(\omega)))ds\\
&\quad+(\omega)\int_{-\infty}^{t}T_{t-s}P^+B_0(s+\tau)dW(s+\tau)-(\omega)\int_t^\infty T_{t-s}P^-B_0(s+\tau)dW(s+\tau)\\
&=\int_{-\infty}^{t+\tau}T_{t+\tau-s}P^+F(s,Y(s,\omega))ds-\int_{t+\tau}^\infty T_{t+\tau-s}P^-F(s,Y(s,\omega))ds\\
&\quad+(\omega)\int_{-\infty}^{t+\tau}T_{t+\tau-s}P^+B_0(s)dW(s)-(\omega)\int_{t+\tau}^\infty T_{t+\tau-s}P^-B_0(s)dW(s)\\
&=\int_{-\infty}^{t}T_{t+\tau-s}P^+F(s,Y(s,\omega))ds-\int_t^\infty T_{t+\tau-s}P^-F(s,Y(s,\omega))ds\\
&\quad+(\omega)\int_{-\infty}^{t}T_{t+\tau-s}P^+B_0(s)dW(s)+(\omega)\int_t^\infty T_{t+\tau-s}P^-B_0(s)dW(s)\\
&\quad+\int_t^{t+\tau}T_{t+\tau-s}P^+F(s,Y(s,\omega))ds+\int_t^{t+\tau}T_{t+\tau-s}P^-F(s,Y(s,\omega))ds\\
&\quad+(\omega)\int_t^{t+\tau}T_{t+\tau-s}P^+B_0(s)dW(s)+(\omega)\int_t^{t+\tau}T_{t+\tau-s}P^-B_0(s)dW(s)\\
&=T_\tau Y(t,\omega)+\int_t^{t+\tau}T_{t+\tau-s}F(s,Y(s,\omega))ds+(\omega)\int_t^{t+\tau}T_{t+\tau-s}B_0(s)dW(s).
\end{aligned}$$

Therefore, $Y(t,\theta_\tau\omega)$, $t\in R$, $\omega\in\Omega$ is a solution of (1.1) with starting point $x=Y(t,\omega)$. Then by the uniqueness of the solution of the initial value problem,

$$u(t+\tau,t,Y(t,\omega),\omega)=Y(t+\tau,\omega)=Y(t,\theta_\tau\omega)$$

for all $t\in R$ and $\omega\in\Omega$.

Conversely, assume equation (1.1) has a random periodic solution which is also tempered from above. First note for any integer m,

$$Y(t,\omega)=u(t\pm m\tau,t,Y(t,\theta_{\mp m\tau}\omega),\theta_{\mp m\tau}\omega)$$

$$= T_{\pm m\tau}Y(t,\theta_{\mp m\tau}\omega) + \int_t^{t\pm m\tau} T_{t\pm m\tau - r}F(r,u(r,t,Y(t,\theta_{\mp m\tau}\omega),\theta_{\mp m\tau}\omega))dr$$
$$+ \int_t^{t\pm m\tau} T_{t\pm m\tau - r}B_0(r)dW(r\mp m\tau).$$

In particular,

$$\begin{aligned}
P^+Y(t,\omega) &= P^+u(t+m\tau,t,Y(t,\theta_{-m\tau}\omega),\theta_{-m\tau}\omega) \\
&= T_{m\tau}P^+Y(t,\theta_{-m\tau}\omega) + \int_t^{t+m\tau} T_{t+m\tau-r}P^+F(r,u(r,t,Y(t,\theta_{-m\tau}\omega),\theta_{-m\tau}\omega))dr \\
&\quad + \int_t^{t+m\tau} T_{t+m\tau-r}P^+B_0(r)dW(r-m\tau) \\
&= T_{m\tau}P^+Y(t,\theta_{-m\tau}\omega) + \int_t^{t+m\tau} T_{t+m\tau-r}P^+F(r-m\tau,Y(r-m\tau,\omega))dr \\
&\quad + \int_t^{t+m\tau} T_{t+m\tau-r}P^+B_0(r-m\tau)dW(r-m\tau) \\
&= T_{m\tau}P^+Y(t,\theta_{-m\tau}\omega) + \int_{t-m\tau}^{t} T_{t-r}P^+F(r,Y(r,\omega))dr + \int_{t-m\tau}^{t} T_{t-r}P^+B_0(r)dW(r) \\
&\to \int_{-\infty}^{t} T_{t-r}P^+F(r,Y(r,\omega))dr + \int_{-\infty}^{t} T_{t-r}P^+B_0(r)dW(r) \qquad (2.4)
\end{aligned}$$

as $m\to\infty$. One can see that the last convergence can be made first in $L^2(dP)$, so

$$P^+Y(t,\omega) = \int_{-\infty}^{t} T_{t-r}P^+F(r,Y(r,\omega))dr + \int_{-\infty}^{t} T_{t-r}P^+B_0(r)dW(r)$$

in $L^2(dP)$, so also a.s. Similarly

$$\begin{aligned}
P^-Y(t,\omega) &= P^-u(t-m\tau,t,Y(t,\theta_{m\tau}\omega),\theta_{m\tau}\omega) \\
&= T_{-m\tau}P^-Y(t,\theta_{m\tau}\omega) - \int_{t-m\tau}^{t} T_{t-m\tau-r}P^-F(r,u(r,t,Y(t,\theta_{m\tau}\omega),\theta_{m\tau}\omega))dr \\
&\quad - \int_{t-m\tau}^{t} T_{t-m\tau-r}P^+B_0(r)dW(r+m\tau) \\
&= T_{-m\tau}P^-Y(t,\theta_{m\tau}\omega) - \int_{t-m\tau}^{t} T_{t-m\tau-r}P^-F(r+m\tau,Y(r+m\tau,\omega))dr \\
&\quad - \int_{t-m\tau}^{t} T_{t-m\tau-r}P^-B_0(r+m\tau)dW(r+m\tau) \\
&= T_{-m\tau}P^-Y(t,\theta_{m\tau}\omega) - \int_{t}^{t+m\tau} T_{t-r}P^-F(r,Y(r,\omega))dr - \int_{t}^{t+m\tau} T_{t-r}P^-B_0(r)dW(r) \\
&\to -\int_{t}^{+\infty} T_{t-r}P^-F(r,Y(r,\omega))dr - \int_{t}^{+\infty} T_{t-r}P^-B_0(r)dW(r) \qquad (2.5)
\end{aligned}$$

as $m\to\infty$. So we have

$$P^-Y(t,\omega) = -\int_{t}^{+\infty} T_{t-r}P^-F(r,Y(r,\omega))dr - \int_{t}^{+\infty} T_{t-r}P^-B_0(r)dW(r), \quad a.s.$$

Therefore we have proved the converse part as $Y = P^+Y + P^-Y$. ♯

Remark 2.1 *Theorem 2.1 also holds in Hilbert space H, but with different assumptions on A and B and W. Assume that A: $D(A) \subset H \to H$ is a closed linear operator and $T_t = e^{-At}$ is the strongly continuous semigroup generated by $-A$. Let E be another separable Hilbert space and $W(t)$, $t \in R$ be an E-valued Brownian motion which is defined on the canonical complete filtered Wiener space $(\Omega, \mathcal{F}, (\mathcal{F}^t)_{t\in R}, P)$ and with covariance in a separable Hilbert space K, where $K \subset E$ is a Hilbert-Schmidt embedding. Here $\mathcal{F}_s^t := \sigma(W_u - W_v, s \leq v \leq u \leq t)$ and $\mathcal{F}^t := \vee_{s\leq t}\mathcal{F}_s^t$. We refer readers to Chapter 4 of Da Prato and Zabczyk [5] for details. Suppose $B_0(s) \in L_2(K,H)$ is a Hilbert Schmidt linear operator with $\sup_{-\infty<s<\infty} ||B_0(s)||_2 < \infty$. Moreover, let A be a self-adjoint operator on H with a discrete non-vanishing spectrum $\{\mu_n,\ n \geq 1\}$ which is bounded below and $\{e_n\}$ be the basis for H consisting of eigenvectors of A. We have $Ae_n = \mu_n e_n$ for $n \geq 1$. Assume further that A^{-1} is trace-class. Denote μ_m the largest negative eigenvalue of A, and μ_{m+1} is its smallest positive eigenvalue. Hence, we obtain an orthogonal splitting of H by two parts. One is $H^- = span\{e_1, e_2, \cdots, e_m\}$ corresponding to the negative eigenvalues $\{\mu_1, \mu_2, \ldots, \mu_m\}$. The other one is $H^+ = span\{e_{m+1}, e_{m+2}, \cdots\}$ corresponding to the positive eigenvalues $\{\mu_n :\ n \geq m+1\}$. And H can be written as*

$$H := H^+ \oplus H^-.$$

We also define the projections onto each subspace by

$$P^+ : H \to H^+,\ P^- : H \to H^-.$$

Since H^- is finite-dimensional, then $T_t|H^-$ on H^- is invertible for each $t \geq 0$. Therefore, we set $T_{-t} := [T_t|H^-]^{-1}$ from $H^- \to H^-$ for each $t \geq 0$. Then everything else discussed above can work the same way.

Before we prove the existence of the equation (2.2), we would like to recall the following standard notation that we will use later. We denote $C_p^\infty(R^n)$ the set of infinitely differentiable functions $f : R^n \to R$ such that f and all its partial derivatives have polynomial growth. Let $\mathcal{S}$ be the class of smooth random variables F that is $F = f(W(h_1), \cdots, W(h_n))$ with $n \in N$, $h_1, \cdots, h_n \in L^2([0,T])$ and $f \in C_p^\infty(R^n)$. The derivative operator of a smooth random variable F is the stochastic process $\{\mathcal{D}_t F,\ t \in [0,T]\}$ defined by (c.f. [18])

$$\mathcal{D}_t F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(W(h_1), \cdots, W(h_n))h_i(t).$$

We will denote $\mathcal{D}^{1,2}$ the domain of $\mathcal{D}$ in $L^2(\Omega)$, i.e. $\mathcal{D}^{1,2}$ is the closure of $\mathcal{S}$ with respect to the norm

$$||F||_{1,2}^2 = E|F|^2 + E||\mathcal{D}_t F||^2_{L^2([0,T])}.$$

Denote $C^0([0,T], L^2(\Omega))$ the set of continuous functions $f(\cdot, \omega)$ with the norm

$$||f||^2 = \sup_{t\in[0,T]} E|f(t,\omega)|^2 < \infty.$$

It's easy to check the following revised version of relative compactness of Wiener-Sobolev space in Bally-Saussereau [2] also holds. This kind of compactness as a purely random variable version without including time and space variables was investigated by Da Prato, Malliavin and Nualart [4] and Peszat [20] first.

Theorem 2.2 *Consider a sequence* $(v_n)_{n\in N}$ *of* $C^0([0,T],L^2(\Omega))$*. Suppose that:*
(1) $v_n(t,\cdot)\in\mathcal{D}^{1,2}$ *and* $\sup_{n\in N}\sup_{t\in[0,T]}||v_n(\cdot,t)||^2_{1,2}<\infty$.
(2) There exists a constant $C>0$ *such that for any* $t_1,t_2\in[0,T]$
$\sup_n E|v_n(t_1)-v_n(t_2)|^2<C|t_1-t_2|$.
(3) (3i) There exists a constant C *such that for any* $0<\alpha<\beta<T$*, and* $h\in R$ *with* $|h|<\min(\alpha,T-\beta)$*, and any* $t_1,t_2\in[0,T]$*,*
$\sup_n\int_\alpha^\beta E|\mathcal{D}_{\theta+h}v_n(t_1)-\mathcal{D}_\theta v_n(t_2)|^2d\theta<C(|h|+|t_1-t_2|)$.
(3ii) For any $\epsilon>0$*, there exist* $0<\alpha<\beta<T$ *such that*
$\sup_n\sup_{t\in[0,T]}\int_{[0,T]\setminus(\alpha,\beta)}E|\mathcal{D}_\theta v_n(t)|^2d\theta<\epsilon$.
Then $\{v_n,n\in N\}$ *is relatively compact in* $C^0([0,T],L^2(\Omega))$*.*

Proof: Recall the Wiener chaos expansion

$$v_n(t,\omega)=\sum_{m=0}^\infty I_m(f_n^m(\cdot,t))(\omega),$$

where $f_n^m(\cdot,t)$ are symmetric elements of $L^2([0,T]^m)$ for each $m\geq 0$. When $m=0$, $f_n^0(t)=Ev_n(t)$, so for any $t_1,t_2\in[0,T]$,

$$\sup_n\sup_{t\in[0,T]}|f_n^0(t)|\leq\sup_n\sup_{t\in[0,T]}\sqrt{E|v_n(t)|^2}<\infty,$$
$$\sup_n|f_n^0(t_1)-f_n^0(t_2)|\leq\sup_n E|v_n(t_1)-v_n(t_2)|\leq\sup_n\sqrt{E|v_n(t_1)-v_n(t_2)|^2}\leq\sqrt{C|t_1-t_2|}.$$

So $\{f_n^0\}_{n=1}^\infty$ is relatively compact in $C^0([0,T])$. For each $m\geq 1$, using the same argument as in Bally-Saussereau [2], we conclude for each fixed t, $\{f_n^m(\cdot,t)\}_{n\in N}$ is relatively compact in $L^2([0,T]^m)$. Moreover, for each $t_1,t_2\in[0,T]$, consider

$$\sup_n||f_n^m(\cdot,t_1)-f_n^m(\cdot,t_2)||^2_{L^2([0,T]^m)}\leq\sup_n\int_0^T E|\mathcal{D}_\theta v_n(t_1)-\mathcal{D}_\theta v_n(t_2)|^2d\theta\leq C|t_1-t_2|,$$

and

$$\sup_n\sup_{t\in[0,T]}||f_n^m(\cdot,t)||^2_{L^2([0,T]^m)}\leq\sup_n\sup_{t\in[0,T]}\int_0^T E|\mathcal{D}_\theta v_n(t)|^2d\theta<\infty.$$

Then by Arzela-Ascoli lemma, we know that $\{f_n^m\}_{n=1}^\infty$ is relatively compact in $C^0([0,T],L^2([0,T]^m))$. Thus we can conclude $\{v_n\}_{n=1}^\infty$ is relatively compact in $C^0([0,T],L^2(\Omega))$ using the same argument as in [2]. ♯

We also need the following generalized Schauder's fixed point theorem to prove our theorem. The proof is refined from the proof of Schauder's fixed point theorem. So we don't claim complete originality but include it here for completeness. Note here we don't require the subset S of Banach space H to be closed, but impose T to be continuous from H to H as the fixed point may not be in S.

Theorem 2.3 *(Generalized Schauder's fixed point theorem) Let* H *be a Banach space,* S *be a convex subset of* H*. Assume a map* $T:H\to H$ *is continuous and* $T(S)\subset S$ *is relatively compact in* H*. Then* T *has a fixed point in* H*.*

Proof: Because $T(S)$ is relatively compact in H and H is a Banach space, so for any $n \in N$, there exists finite $\frac{1}{n}$ - net $N_n = \{y_1, y_2, \cdots, y_{r_n}\}$ such that

$$T(S) \subset \bigcup_{i=1}^{r_n} B(y_i, \frac{1}{n}),$$

where $B(y_i, \frac{1}{n}) = \{y : ||y - y_i||_H < \frac{1}{n}\}$, $y_i \in T(S)$, $i = 1, \cdots, r_n$. Denote $E_n := span\{N_n\}$, the finite dimensional linear subspace spanned by N_n.

Define a map $I_n : T(S) \to co(N_n)$ by

$$I_n(y) = \sum_{i=1}^{r_n} y_i \lambda_i(y), \tag{2.6}$$

where $co(N_n)$ is the all convex combination of the elements in N_n, and

$$\lambda_i(y) = \frac{m_i(y)}{\sum\limits_{i=1}^{r_n} m_i(y)}, \quad m_i(y) = \begin{cases} 1 - n||y - y_i||_H, & \text{if } y \in B(y_i, \frac{1}{n}), \\ 0, & \text{if } y \notin B(y_i, \frac{1}{n}). \end{cases}$$

It's easy to see that $m_i(y) \geq 0$, and for any $y \in T(S)$, there exists an i_0 $(1 \leq i_0 \leq r_n)$ such that $y \in B(y_{i_0}, \frac{1}{n})$, so $m_{i_0}(y) > 0$. Therefore $\lambda_i(y)$ $(1 \leq i \leq r_n)$ can be defined and satisfy

$$\lambda_i(y) \geq 0 \ (1 \leq i_0 \leq r_n), \quad \sum_{i=1}^{r_n} \lambda_i(y) = 1. \tag{2.7}$$

So I_n can be defined on $T(S)$ and from (2.6) and (2.7) we can see $I_n(y)$ is the convex combination of the elements in N_n. Hence, $I_n(y) \in co(N_n)$. Moreover, for any $y \in T(S)$,

$$\begin{aligned} ||I_n(y) - y||_H &= ||\sum_{i=1}^{r_n} y_i \lambda_i(y) - \sum_{i=1}^{r_n} y \lambda_i(y)||_H \\ &\leq \sum_{i=1}^{r_n} ||y_i - y||_H \lambda_i(y) \\ &= \sum_{y \in B(y_i, \frac{1}{n})}^{r_n} ||y_i - y||_H \lambda_i(y) + \sum_{y \notin B(y_i, \frac{1}{n})}^{r_n} ||y_i - y||_H \lambda_i(y) \\ &< \frac{1}{n}. \end{aligned} \tag{2.8}$$

Note that $T(S) \subset S$, $N_n \subset T(S)$ and S is convex, so $co(N_n) \subset S$. Define $T_n := I_n \circ T$. Then $T_n : co(N_n) \to co(N_n)$. But $co(N_n)$ is a bounded closed convex subset in E_n, so by the Brouwer's fixed point theorem, there exists $x_n \in co(N_n) \subset S$ such that

$$T_n x_n = x_n. \tag{2.9}$$

On the other hand, $T(S)$ is relatively compact in H and H is complete, so there exists a subsequence $\{x_{n_k}\} \in S$ and $x \in H$ such that

$$T x_{n_k} \to x, \quad as \ k \to \infty. \tag{2.10}$$

From (2.8) and (2.9), we have

$$
\begin{aligned}
||x_n - x||_H &= ||T_n x_n - x||_H \\
&\le ||I_n T x_n - T x_n||_H + ||T x_n - x||_H \\
&< \frac{1}{n} + ||T x_n - x||_H .
\end{aligned} \tag{2.11}
$$

Combining (2.10) and (2.11), we can get $x_{n_k} \to x$, as $k \to \infty$. As T is continuous and also from (2.10), we have

$$Tx = x. \qquad \sharp$$

Now we are going to prove that equation (2.2) has a solution under some conditions. So according to Theorem 2.1, this gives the existence of the random periodic solution for the stochastic evolution equation (1.1).

Theorem 2.4 *Assume above conditions on A and B_0. Let $F : (-\infty, \infty) \times R^d \to R^d$ be a continuous map, globally bounded and the Jacobian $\nabla F(t, \cdot)$ be globally bounded, and F and B_0 also satisfy Condition (P) and there exists a constant $L_1 > 0$ such that $||B_0(s_1) - B_0(s_2)|| \le L_1 |s_1 - s_2|^{\frac{1}{2}}$. Then there exists at least one $\mathcal{B}(R) \otimes \mathcal{F}$-measurable map $Y : (-\infty, +\infty) \times \Omega \to R^d$ satisfying equation (2.2) and $Y(t + \tau, \omega) = Y(t, \theta_\tau \omega)$ for any $t \in R$ and $\omega \in \Omega$.*

Proof: Firstly, define the $\mathcal{B}(R) \otimes \mathcal{F}$-measurable map $Y_1 : (-\infty, +\infty) \times \Omega \to R^d$ by

$$Y_1(t, \omega) = (\omega) \int_{-\infty}^{t} T_{t-s} P^+ B_0(s) dW(s) - (\omega) \int_{t}^{\infty} T_{t-s} P^- B_0(s) dW(s). \tag{2.12}$$

Then we have

$$
\begin{aligned}
Y_1(t, \theta_\tau \omega) &= (\theta_\tau \omega) \int_{-\infty}^{t} T_{t-s} P^+ B_0(s) dW(s) - (\theta_\tau \omega) \int_{t}^{\infty} T_{t-s} P^- B_0(s) dW(s) \\
&= (\omega) \int_{-\infty}^{t+\tau} T_{t+\tau-s} P^+ B_0(s) dW(s) - (\omega) \int_{t+\tau}^{\infty} T_{t+\tau-s} P^- B_0(s) dW(s) \\
&= Y_1(t + \tau, \omega).
\end{aligned} \tag{2.13}
$$

Secondly, we need to solve the equation

$$Z(t, \omega) = \int_{-\infty}^{t} T_{t-s} P^+ F(s, Z(s, \omega) + Y_1(s, \omega))) ds - \int_{t}^{\infty} T_{t-s} P^- F(s, Z(s, \omega) + Y_1(s, \omega))) ds. \tag{2.14}$$

We will do this in the following several steps.

Step 1 : Define

$$C_\tau^0((-\infty, +\infty), L^2(\Omega)) := \{ f \in C^0((-\infty, +\infty), L^2(\Omega)) : \text{ for any } \ t \in (-\infty, \infty), \ f(\tau + t, \omega) = f(t, \theta_\tau \omega) \}.$$

For any $z \in C_\tau^0((-\infty, +\infty), L^2(\Omega))$, define

$$\mathcal{M}(z)(t, \omega) = \int_{-\infty}^{t} T_{t-s} P^+ F(s, z(s, \omega) + Y_1(s, \omega)) ds - \int_{t}^{\infty} T_{t-s} P^- F(s, z(s, \omega) + Y_1(s, \omega)) ds.$$

We will prove $\mathcal{M}$ maps $C_\tau^0((-\infty, +\infty), L^2(\Omega))$ to itself. Firstly, $\mathcal{M}(z)(\cdot, \omega)$ is continuous. For this, taking any t_1, $t_2 \in (-\infty, +\infty)$ with $t_1 \le t_2$, we have

$$
\begin{aligned}
&E|\mathcal{M}(z)(t_1)-\mathcal{M}(z)(t_2)|^2\\
\leq\ & 2E\Big[\Big|\int_{-\infty}^{t_1} T_{t_1-s}P^+F(s,z(s,\omega)+Y_1(s,\omega))ds-\int_{-\infty}^{t_2} T_{t_2-s}P^+F(s,z(s,\omega)+Y_1(s,\omega))ds\Big|^2\\
&+\Big|\int_{t_2}^{+\infty} T_{t_2-s}P^-F(s,z(s,\omega)+Y_1(s,\omega))ds-\int_{t_2}^{+\infty} T_{t_2-s}P^-F(s,z(s,\omega)+Y_1(s,\omega))ds\Big|^2\Big].
\end{aligned}
$$

For the first term, we have the following estimate,

$$
\begin{aligned}
&E\Big|\int_{-\infty}^{t_1} T_{t_1-s}P^+F(s,z(s,\omega)+Y_1(s,\omega))ds-\int_{-\infty}^{t_2} T_{t_2-s}P^+F(s,z(s,\omega)+Y_1(s,\omega))ds\Big|^2\\
\leq\ & 2E\Big|\int_{-\infty}^{t_1}(T_{t_1-s}P^+-T_{t_2-s}P^+)F(s,z(s,\omega)+Y_1(s,\omega))ds\Big|^2\\
&+2E\Big|\int_{t_1}^{t_2} T_{t_2-s}P^+F(s,z(s,\omega)+Y_1(s,\omega))ds\Big|^2\\
\leq\ & 2||F||_\infty^2\Big[\Big(\int_{-\infty}^{t_1}\parallel T_{t_1-s}P^+-T_{t_2-s}P^+\parallel ds\Big)^2+\Big(\int_{t_1}^{t_2}\parallel T_{t_2-s}P^+\parallel ds\Big)^2\Big]\\
\leq\ & 2||F||_\infty^2\Big[\Big(\parallel I-T_{t_2-t_1}P^+\parallel\int_{-\infty}^{t_1} e^{-(t_1-s)\mu_{m+1}}ds\Big)^2+\Big(\int_{t_1}^{t_2} \mathrm{e}^{-(t_2-s)\mu_{m+1}}ds\Big)^2\Big]\\
\leq\ & 4||F||_\infty^2|t_2-t_1|^2.
\end{aligned}
$$

Here

$$
||F||_\infty:=\sup_{t\in(-\infty,+\infty),u\in R^d}|F(t,u)|.
$$

And by a similar argument to the second part, we have

$$
\begin{aligned}
&E\Big|\int_{t_1}^{+\infty} T_{t_1-s}P^-F(s,z(s,\omega)+Y_1(s,\omega))ds-\int_{t_2}^{+\infty} T_{t_2-s}P^-F(s,z(s,\omega)+Y_1(s,\omega))ds\Big|^2\\
\leq\ & 2E\Big|\int_{t_2}^{+\infty}(T_{t_1-s}P^--T_{t_2-s}P^-)F(s,z(s,\omega)+Y_1(s,\omega))ds\Big|^2\\
&+2E\Big|\int_{t_1}^{t_2} T_{t_1-s}P^-F(s,z(s,\omega)+Y_1(s,\omega))ds\Big|^2\\
\leq\ & 2||F||_\infty^2\Big[\Big(\parallel T_{t_1-t_2}P^--I\parallel\int_{t_2}^{\infty} e^{-(t_2-s)\mu_m}ds\Big)^2+\Big(\int_{t_1}^{t_2} \mathrm{e}^{-(t_1-s)\mu_m}ds\Big)^2\Big]\\
\leq\ & 4||F||_\infty^2|t_2-t_1|^2.
\end{aligned}
$$

Therefore, by combining two parts, we have

$$
E|\mathcal{M}(z)(t_1)-\mathcal{M}(z)(t_2)|^2\leq 8||F||_\infty^2|t_2-t_1|^2.
$$

Secondly,

$$
\begin{aligned}
E|\mathcal{M}(z)(t,\omega)|^2\leq\ & 2E\Big|\int_{-\infty}^{t} T_{t-s}P^+F(s,z(s,\omega)+Y_1(s,\omega))ds\Big|^2\\
&+2E\Big|\int_{t}^{+\infty} T_{t-s}P^-F(s,z(s,\omega)+Y_1(s,\omega))ds\Big|^2\\
\leq\ & 2E\int_{-\infty}^{t}|T_{t-s}P^+|ds\cdot\int_{-\infty}^{t}|T_{t-s}P^+|\cdot|F(s,z(s,\omega)+Y_1(s,\omega))|^2ds
\end{aligned}
$$

$$
\begin{aligned}
&+2E\int_t^{+\infty}|T_{t-s}P^-|ds\cdot\int_t^{+\infty}|T_{t-s}P^-|\cdot|F(s,z(s,\omega)+Y_1(s,\omega))|^2ds\\
&\leq 2||F||_\infty^2\Big[\Big(\int_{-\infty}^t|T_{t-s}P^+|ds\Big)^2+\Big(\int_t^{+\infty}|T_{t-s}P^-|ds\Big)^2\Big]\\
&\leq 2||F||_\infty^2\Big[\Big(\int_{-\infty}^t e^{-(t-s)\mu_{m+1}}ds\Big)^2+\Big(\int_t^{+\infty}e^{-(t-s)\mu_m}ds\Big)^2\Big]\\
&\leq 2||F||_\infty^2(\frac{1}{\mu_{m+1}^2}+\frac{1}{\mu_m^2}).
\end{aligned}
$$

So

$$
||\mathcal{M}(z)(t,\omega)||^2=\sup_{t\in(-\infty,+\infty)}E|\mathcal{M}(z)(t,\omega)|^2\leq 2||F||_\infty^2(\frac{1}{\mu_{m+1}^2}+\frac{1}{\mu_m^2})<\infty.
$$

Thirdly,

$$
\begin{aligned}
\mathcal{M}(z)(t,\theta_\tau\omega)&=\int_{-\infty}^t T_{t-s}P^+F(s+\tau,z(s+\tau,\omega)+Y_1(s,\theta_\tau\omega))ds\\
&\quad-\int_t^{+\infty}T_{t-s}P^-F(s+\tau,z(s+\tau,\omega)+Y_1(s,\theta_\tau\omega))ds\\
&=\int_{-\infty}^{t+\tau}T_{t+\tau-s}P^+F(s,z(s,\omega)+Y_1(s,\omega))ds\\
&\quad-\int_{t+\tau}^{+\infty}T_{t+\tau-s}P^-F(s,z(s,\omega)+Y_1(s,\omega))ds\\
&=\mathcal{M}(z)(t+\tau,\omega).
\end{aligned}
$$

Therefore, we can see $\mathcal{M}$ maps $C_\tau^0((-\infty,+\infty),L^2(\Omega))$ into itself.

Step 2: To see the continuity, for any $z_1,z_2\in C_\tau^0((-\infty,+\infty),L^2(\Omega))$,

$$
\begin{aligned}
&\sup_{t\in(-\infty,+\infty)}E|\mathcal{M}(z_1)(t,\omega)-\mathcal{M}(z_2)(t,\omega)|^2\\
&\leq||\nabla F||_\infty^2\sup_{t\in(-\infty,+\infty)}E\int_{-\infty}^t||T_{t-s}P^+||ds\int_{-\infty}^t||T_{t-s}P^+||\cdot|z_1(s,\omega)-z_2(s,\omega)|^2ds\\
&\quad+||\nabla F||_\infty^2\sup_{t\in(-\infty,+\infty)}E\int_t^{+\infty}||T_{t-s}P^-||ds\int_t^{+\infty}||T_{t-s}P^-||\cdot|z_1(s,\omega)-z_2(s,\omega)|^2ds\\
&\leq||\nabla F||_\infty^2(\frac{1}{\mu_{m+1}^2}+\frac{1}{\mu_m^2})\sup_{t\in(-\infty,+\infty)}E|z_1(t,\omega)-z_2(t,\omega)|^2,
\end{aligned}
$$

where

$$
||\nabla F||_\infty:=\sup_{t\in(-\infty,\infty),u\in R^d}||\nabla F(t,u)||_{\mathcal{L}(R^d)}.
$$

That is to say that $\mathcal{M}:C_\tau^0((-\infty,+\infty),L^2(\Omega))\to C_\tau^0((-\infty,+\infty),L^2(\Omega))$ is a continuous map.

Step 3: Now let's define a subset of $C_\tau^0((-\infty,+\infty),L^2(\Omega))$ as follows:

$$
\begin{aligned}
C_{\tau,\alpha}^0((-\infty,+\infty),\mathcal{D}^{1,2}):=\{\ &f\in C_\tau^0((-\infty,+\infty),L^2(\Omega)):\ f|_{[0,\tau)}\in C^0([0,\tau),\mathcal{D}^{1,2}), \qquad (2.15)\\
&i.e.\ ||f||^2=\sup_{t\in[0,\tau)}||f(t,\omega)||_{1,2}^2<\infty,\ \text{and for any } t,r\in[0,\tau),\\
&E|\mathcal{D}_rf(t,\omega)|^2\leq 2\tau^2||\nabla F||_\infty^2\cdot||B_0||_\infty^2\cdot\exp\{2\tau||\nabla F||_\infty^2|t-r|)\}:=\alpha(t,r),\\
&\sup_{s,r_1,r_2\in[0,\tau)}\frac{E|\mathcal{D}_{r_1}f(s,\omega)-\mathcal{D}_{r_2}f(s,\omega)|^2}{|r_1-r_2|}<\infty\}. \qquad (2.16)
\end{aligned}
$$

This is a convex set.
(I) We will prove that $\mathcal{M}$ maps $C^0_{\tau,\alpha}((-\infty,+\infty),\mathcal{D}^{1,2})$ into itself. The Malliavin derivatives of $Y_1(t,\omega)$ and $\mathcal{M}(z)(t,\omega)$ can be calculated as:

$$\mathcal{D}_r Y_1(t,\omega)=\begin{cases} T_{t-r}P^+B_0(r), & \text{if } r\le t,\\ -T_{t-r}P^-B_0(r), & \text{if } r>t.\end{cases} \tag{2.17}$$

$$\begin{aligned}&\mathcal{D}_r\mathcal{M}(z)(t,\omega)\\ =&\begin{cases}\int_r^t T_{t-s}P^+\nabla F(s,z(s,\omega)+Y_1(s,\omega))(\mathcal{D}_r z(s,\omega)+T_{s-r}P^+B_0(r))ds, & \text{if } r\le t,\\ -\int_t^r T_{t-s}P^-\nabla F(s,z(s,\omega)+Y_1(s,\omega))(\mathcal{D}_r z(s,\omega)-T_{s-r}P^-B_0(r))ds, & \text{if } r>t.\end{cases}\end{aligned} \tag{2.18}$$

So for any $z\in C^0_{\tau,\alpha}((-\infty,+\infty),\mathcal{D}^{1,2})$, when $0\le r\le t<\tau$,

$$\begin{aligned}&E|\mathcal{D}_r\mathcal{M}(z)(t,\omega)|^2\\ \le&\ 2E\int_r^t|T_{t-s}P^+\nabla F(s,z(s,\omega)+Y_1(s,\omega))|^2ds\cdot\int_r^t|\mathcal{D}_r z(s,\omega)|^2ds\\ &+2E\Big[\int_r^t T_{t-r}P^+\nabla F(s,z(s,\omega)+Y_1(s,\omega))B_0(r)ds\Big]^2\\ \le&\ 2\tau||\nabla F||^2_\infty\int_r^t 2\tau^2||\nabla F||^2_\infty\cdot||B_0||^2_\infty\cdot\exp\{2\tau||\nabla F||^2_\infty(s-r)\}ds+2\tau^2||\nabla F||^2_\infty||B_0||^2_\infty\\ =&\ 2\tau^2||\nabla F||^2_\infty\cdot||B_0||^2_\infty\cdot\exp\{2\tau||\nabla F||^2_\infty(t-r)\}\\ =&\ \alpha(t,r).\end{aligned}$$

Similarly, when $0\le t<r<\tau$,

$$\begin{aligned}&E|\mathcal{D}_r\mathcal{M}(z)(t,\omega)|^2\\ \le&\ 2E\int_t^r|T_{t-s}P^-\nabla F(s,z(s,\omega)+Y_1(s,\omega))|^2ds\cdot\int_t^r|\mathcal{D}_r z(s,\omega)|^2ds\\ &+2E\Big[\int_t^r T_{t-r}P^-\nabla F(s,z(s,\omega)+Y_1(s,\omega))B_0(r)ds\Big]^2\\ \le&\ 2\tau||\nabla F||^2_\infty\int_t^r 2\tau^2||\nabla F||^2_\infty\cdot||B_0||^2_\infty\cdot\exp\{2\tau||\nabla F||^2_\infty(r-s)\}ds+2\tau^2||\nabla F||^2_\infty||B_0||^2_\infty\\ =&\ 2\tau^2||\nabla F||^2_\infty\cdot||B_0||^2_\infty\cdot\exp\{2\tau||\nabla F||^2_\infty(r-t)\}\\ =&\ \alpha(t,r).\end{aligned}$$

Therefore, we have

$$E|\mathcal{D}_r\mathcal{M}(z)(t,\omega)|^2\le\alpha(t,r).$$

Moreover, because $z\in C^0_{\tau,\alpha}((-\infty,+\infty),\mathcal{D}^{1,2})$, we have

$$E|\mathcal{D}_r z(t,\omega)|^2\le 2\tau^2||\nabla F||^2_\infty||B_0||^2_\infty\exp\{2\tau^2||\nabla F||^2_\infty\}:=\alpha_1.$$

and also

$$E|\mathcal{D}_r\mathcal{M}(z)(t,\omega)|^2\le\alpha_1.$$

Suppose that for any $r_1,r_2,s\in[0,\tau)$, there exists $L_2\ge 0$ such that

$$\frac{E|\mathcal{D}_{r_1}z(s,\omega)-\mathcal{D}_{r_2}z(s,\omega)|^2}{|r_1-r_2|}\leq L_2.$$

Then we have when $0\leq r_1<r_2\leq t<\tau$,

$$\begin{aligned}
&\frac{1}{|r_1-r_2|}E|\mathcal{D}_{r_1}\mathcal{M}(z)(t,\omega)-\mathcal{D}_{r_2}\mathcal{M}(z)(t,\omega)|^2\\
\leq\ &\frac{2}{|r_1-r_2|}\Big[E\Big|\int_{r_1}^{t}T_{t-s}P^+\nabla F(s,z(s,\omega)+Y_1(s,\omega))(\mathcal{D}_{r_1}z(s,\omega)-\mathcal{D}_{r_2}z(s,\omega)\\
&\qquad\qquad +T_{s-r_1}P^+B_0(r_1)-T_{s-r_2}P^+B_0(r_2))ds\Big|^2\\
&\qquad +E\Big|\int_{r_1}^{r_2}T_{t-s}P^+\nabla F(s,z(s,\omega)+Y_1(s,\omega))(\mathcal{D}_{r_2}z(s,\omega)+T_{s-r_2}P^+B_0(r_2))ds\Big|^2\Big]\\
\leq\ &4||\nabla F||_\infty^2L_2\int_{r_1}^{t}||T_{t-s}P^+||^2ds\\
&+\frac{4}{|r_1-r_2|}E\Big|\int_{r_1}^{t}\nabla F(s,z(s,\omega)+Y_1(s,\omega))(T_{t-r_1}P^+B_0(r_1)-T_{t-r_1}P^+B_0(r_2)\\
&\qquad\qquad +T_{t-r_1}P^+B_0(r_2)-T_{t-r_2}P^+B_0(r_2))ds\Big|^2\\
&+4||\nabla F||_\infty^2(E\int_{r_1}^{r_2}||T_{t-s}P^+||^2|\nabla F(s,z(s,\omega)+Y_1(s,\omega))|^2ds\int_{r_1}^{r_2}|\mathcal{D}_{r_2}z(s,\omega)|^2ds\\
&\qquad +|r_2-r_1|^2||T_{t-r_2}P^+||^2||B_0||_\infty^2)\\
\leq\ &||\nabla F||_\infty^2\frac{2L_2}{\mu_{m+1}}\\
&+\frac{8}{|r_1-r_2|}||\nabla F||_\infty^2(t-r_1)^2(e^{-2\mu_{m+1}(t-r_1)}L_1^2(r_2-r_1)+||B_0||_\infty^2e^{-2\mu_{m+1}(t-r_2)}||I-T_{r_2-r_1}P^+||^2)\\
&+4||\nabla F||_\infty^2(\alpha_1+||B_0||_\infty^2)(r_2-r_1)\\
\leq\ &2||\nabla F||_\infty^2\Big[\frac{L_2}{\mu_{m+1}}+4\tau^2L_1^2+(4\tau^2||B_0||_\infty^2\mu_{m+1}^2+2\alpha_1+2||B_0||_\infty^2)(r_2-r_1)\Big]\\
:=\ &C.
\end{aligned}$$

When $0\leq r_1<t<r_2<\tau$

$$\begin{aligned}
&\frac{1}{|r_1-r_2|}E|\mathcal{D}_{r_1}\mathcal{M}(z)(t,\omega)-\mathcal{D}_{r_2}\mathcal{M}(z)(t,\omega)|^2\\
\leq\ &\frac{4}{|r_1-r_2|}||\nabla F||_\infty^2(t-r_1)E\Big[\int_{r_1}^{t}|T_{t-s}P^+|^2|\mathcal{D}_{r_1}z(s,\omega)|^2ds+\int_{r_1}^{t}|T_{t-r_1}P^+|^2|B_0(r_1))|^2ds\Big]\\
&+\frac{4}{|r_1-r_2|}||\nabla F||_\infty^2(r_2-t)E\Big[\int_{t}^{r_2}|T_{t-s}P^-|^2|\mathcal{D}_{r_2}z(s,\omega)|^2ds+\int_{t}^{r_2}|T_{t-r_2}P^-|^2|B_0(r_2))|^2ds\Big]\\
\leq\ &\frac{4}{|r_1-r_2|}||\nabla F||_\infty^2(\alpha_1+||B_0||_\infty^2)((t-r_1)^2+(r_2-t)^2)\\
=\ &||\nabla F||_\infty^2(\alpha_1+||B_0||_\infty^2)|r_2-r_1|\\
:=\ &\tilde{C}.
\end{aligned}$$

When $0\leq t\leq r_1<r_2<\tau$,

$$\frac{1}{|r_1-r_2|}E|\mathcal{D}_{r_1}\mathcal{M}(z)(t,\omega)-\mathcal{D}_{r_2}\mathcal{M}(z)(t,\omega)|^2$$

$$\leq 2||\nabla F||_\infty^2\Big[-\frac{L_2}{\mu_m}+4\tau^2L_1^2+(4\tau^2||B_0||_\infty^2\mu_m^2+2\alpha_1+2||B_0||_\infty^2)(r_2-r_1)\Big]$$
$$:=\hat{C}.$$

Therefore, $\mathcal{M}$ maps $C^0_{\tau,\alpha}((-\infty,+\infty),\mathcal{D}^{1,2})$ to itself.
(II) Define

$$\mathcal{M}(C^0_{\tau,\alpha}((-\infty,+\infty),\mathcal{D}^{1,2}))|_{[0,\tau)}:=\{f|_{[0,\tau)}:f\in\mathcal{M}(C^0_{\tau,\alpha}((-\infty,+\infty),\mathcal{D}^{1,2}))\},$$

we will prove $\mathcal{M}(C^0_{\tau,\alpha}((-\infty,+\infty),\mathcal{D}^{1,2}))|_{[0,\tau)}$ is relatively compact in $C^0([0,\tau),L^2(\Omega))$. With what we have proved in (I), we also need to prove $\mathcal{D}_r\mathcal{M}(z)(t,\omega)$ is equicontinuous in t. We will consider several cases.
When $0\leq r<t_1<t_2<\tau$,

$$\begin{aligned}
&E|\mathcal{D}_r\mathcal{M}(z)(t_1,\omega)-\mathcal{D}_r\mathcal{M}(z)(t_2,\omega)|^2\\
\leq\ &2E\Big|\int_r^{t_1}(T_{t_1-s}P^+-T_{t_2-s}P^+)\nabla F(s,z(s,\omega)+Y_1(s,\omega))(\mathcal{D}_rz(s,\omega)+T_{s-r}P^+B_0(r))ds\Big|^2\\
&+2E\Big|\int_{t_1}^{t_2}T_{t_2-s}P^+\nabla F(s,z(s,\omega)+Y_1(s,\omega))(\mathcal{D}_rz(s,\omega)+T_{s-r}P^+B_0(r))ds\Big|^2\\
\leq\ &4||\nabla F||_\infty^2\Big[\Big(\int_r^{t_1}|T_{t_1-s}P^+-T_{t_2-s}P^+|ds\Big)^2\cdot\int_r^{t_1}E|\mathcal{D}_rz(s,\omega)|^2ds\\
&\qquad+||B_0||_\infty^2(t_1-r)^2|T_{t_1-r}P^+-T_{t_2-r}P^+|^2\\
&\qquad+\int_{t_1}^{t_2}|T_{t_2-s}P^+|^2ds\cdot\int_{t_1}^{t_2}E|\mathcal{D}_rz(s,\omega)|^2ds\\
&\qquad+||B_0||_\infty^2(t_2-t_1)^2|T_{t_2-r}P^+|^2\Big]\\
\leq\ &4||\nabla F||_\infty^2\Big[\alpha_1||I-T_{t_2-t_1}P^+||^2\Big(\int_r^{t_1}e^{-(t_1-s)\mu_{m+1}}ds\Big)^2\\
&\qquad+(t_1-r)^2e^{-2(t_1-r)\mu_{m+1}}\cdot||I-T_{t_2-t_1}P^+||^2\cdot||B_0||_\infty^2\\
&\qquad+\Big(\alpha_1+||B_0||_\infty^2\Big)(t_2-t_1)^2\Big]\\
\leq\ &8||\nabla F||_\infty^2\Big(\alpha_1+||B_0||_\infty^2\Big)(t_2-t_1)^2.
\end{aligned}$$

When $0\leq t_1<r<t_2<\tau$,

$$\begin{aligned}
&E|\mathcal{D}_r\mathcal{M}(z)(t_2,\omega)-\mathcal{D}_r\mathcal{M}(z)(t_1,\omega)|^2\\
\leq\ &4||\nabla F||_\infty^2\Big[\int_r^{t_2}||T_{t_2-s}P^+||^2ds\int_r^{t_2}E|\mathcal{D}_rz(s,\omega)|^2ds+(t_2-r)^2||B_0||_\infty^2||T_{t_2-r}P^+||^2\\
&\qquad+\int_{t_1}^{r}||T_{t_1-s}P^-||^2ds\int_{t_1}^{r}E|\mathcal{D}_rz(s,\omega)|^2ds+(r-t_1)^2||B_0||_\infty^2||T_{t_1-r}P^-||^2\Big]\\
\leq\ &8||\nabla F||_\infty^2\Big(\alpha_1+||B_0||_\infty^2\Big)(t_2-t_1)^2.
\end{aligned}$$

When $0\leq t_1<t_2<r<\tau$,

$$\begin{aligned}
&E|\mathcal{D}_r\mathcal{M}(z)(t_2,\omega)-\mathcal{D}_r\mathcal{M}(z)(t_1,\omega)|^2\\
\leq\ &2E\Big|\int_{t_2}^{r}(T_{t_1-s}P^--T_{t_2-s}P^-)\nabla F(s,z(s,\omega)+Y_1(s,\omega))(\mathcal{D}_rz(s,\omega)-T_{s-r}P^-B_0(r))ds\Big|^2
\end{aligned}$$

$$
\begin{aligned}
&+2E\Big|\int_{t_1}^{t_2} T_{t_1-s}P^-\nabla F(s,z(s,\omega)+Y_1(s,\omega))(\mathcal{D}_r z(s,\omega)-T_{s-r}P^-B_0(r))ds\Big|^2\\
\le\ & 4||\nabla F||_\infty^2\Big[\alpha_1||T_{t_2-t_1}P^- - I||^2\Big(\int_{t_2}^{r} e^{-(t_2-s)\mu_m}ds\Big)^2\\
&\quad +(r-t_2)^2 e^{-2(r-t_2)\mu_m}\cdot||T_{t_2-t_1}P^- - I||^2\cdot||B_0||_\infty^2\\
&\quad +\Big(\alpha_1+||B_0||_\infty^2\Big)(t_2-t_1)^2\Big]\\
\le\ & 8||\nabla F||_\infty^2\Big(\alpha_1+||B_0||_\infty^2\Big)(t_2-t_1)^2.
\end{aligned}
$$

Thus, from the above arguments, by Theorem 2.2, $\mathcal{M}(C^0_{\tau,\alpha}((-\infty,+\infty),\mathcal{D}^{1,2}))|_{[0,\tau)}$ is relatively compact in $C^0([0,\tau),L^2(\Omega))$.
(III) We need to prove that $\mathcal{M}(C^0_{\tau,\alpha}((-\infty,+\infty),\mathcal{D}^{1,2}))$ is relatively compact in $C^0_\tau((-\infty,+\infty),L^2(\Omega))$. From (II), we know for any sequence $\mathcal{M}(z_n)\in C^0_{\tau,\alpha}((-\infty,+\infty),\mathcal{D}^{1,2})$, there exists a subsequence, still denoted by $\mathcal{M}(z_n)$ and $Z^*\in C^0([0,\tau),L^2(\Omega))$ such that

$$
\sup_{t\in[0,\tau)} E|\mathcal{M}(z_n)(t,\omega)-Z^*(t,\omega)|^2\to 0 \tag{2.19}
$$

as $n\to\infty$. Set for $\tau\le t<2\tau$,

$$
Z^*(t,\omega)=Z^*(t-\tau,\theta_\tau\omega).
$$

Noting

$$
\mathcal{M}(z_n)(t,\theta_\tau\omega)=\mathcal{M}(z_n)(t+\tau,\omega),
$$

from (2.19), and the probability preserving of θ, we have

$$
\begin{aligned}
&\sup_{t\in[\tau,2\tau)} E|\mathcal{M}(z_n)(t,\omega)-Z^*(t,\omega)|^2\\
=&\sup_{t\in[0,\tau)} E|\mathcal{M}(z_n)(t+\tau,\omega)-Z^*(t+\tau,\omega)|^2\\
=&\sup_{t\in[0,\tau)} E|\mathcal{M}(z_n)(t,\theta_\tau\omega)-Z^*(t,\theta_\tau\omega)|^2\\
=&\sup_{t\in[0,\tau)} E|\mathcal{M}(z_n)(t,\omega)-Z^*(t,\omega)|^2\\
\to&\ 0,
\end{aligned}
$$

Similarly one can prove that

$$
\sup_{t\in[0,\tau)} E|\mathcal{M}(z_n)(t+m\tau,\omega)-Z^*(t+m\tau,\omega)|^2=\sup_{t\in[0,\tau)} E|\mathcal{M}(z_n)(t,\omega)-Z^*(t,\omega)|^2\to 0, \tag{2.20}
$$

for any $m\in\{0,\pm1,\pm2,\cdots\}$. Therefore

$$
\sup_{t\in(-\infty,+\infty)} E|\mathcal{M}(z_n)(t,\omega)-Z^*(t,\omega)|^2\to 0,
$$

as $n\to\infty$. Therefore $\mathcal{M}(C^0_{\tau,\alpha}((-\infty,+\infty),\mathcal{D}^{1,2}))$ is relatively compact in $C^0_\tau((-\infty,+\infty),L^2(\Omega))$.

Stpe 4: According to the generalized Schauder's fixed point theorem, $\mathcal{M}$ has a fixed point in $C^0_\tau((-\infty,+\infty),L^2(\Omega))$. That is to say there exists a solution $Z\in C^0_\tau((-\infty,+\infty),L^2(\Omega))$of equation (2.14) such that for any $t\in(-\infty,+\infty)$, $Z(t+\tau,\omega)=Z(t,\theta_\tau\omega)$. Then $Y=Z+Y_1$ is the desired solution of (2.2). Moreover, $Y(t+\tau,\omega)=Y(t,\theta_\tau\omega)$. ♯

Now we consider the semilinear stochastic differential equations with the additive noise of the form

$$\begin{aligned} du(t) &= [-Au(t)+F(u(t))]dt+B_0dW(t), \\ u(0) &= x\in R^d, \end{aligned} \tag{2.21}$$

for $t\geq 0$. Here F and B_0 do not depend on time t, that is to say, τ in Condition (P) can be chosen as an arbitrary real number. We have a similar variation of constant representation to (2.2). The difference is that for this equation, we have a cocycle. Similar to Theorem 2.1, we can prove the following theorem without giving the proof here.

Theorem 2.5 *Assume Cauchy problem (2.21) has a unique solution $u(t,x,\omega)$ and the coupled forward-backward infinite horizon stochastic integral equation*

$$\begin{aligned} Y(\omega) &= \int_{-\infty}^0 T_{-s}P^+F(Y(\theta_s\omega))ds-\int_0^\infty T_{-s}P^-F(Y(\theta_s\omega))ds \\ &+(\omega)\int_{-\infty}^0 T_{-s}P^+B_0dW(s)-(\omega)\int_0^\infty T_{-s}P^-B_0dW(s) \end{aligned} \tag{2.22}$$

has one solution $Y:\Omega\to R^d$, then Y is a stationary solution of equation (2.21) i.e.

$$u(t,Y(\omega),\omega)=Y(\theta_t\omega) \quad \text{for any} \quad t\geq 0 \qquad a.s. \tag{2.23}$$

Conversely, if equation (2.21) has a stationary solution $Y:\Omega\to R^d$ which is tempered from above, then Y is a solution of the coupled forward-backward infinite horizon stochastic integral equation (2.22).

Theorem 2.6 *Assume the above conditions on A and B_0. Let $F:R^d\to R^d$ be a continuous map, globally bounded and ∇F be globally bounded. Then there exists at least one $\mathcal{F}$-measurable map $Y:\Omega\to R^d$ satisfying (2.22).*

Proof: Set the $\mathcal{F}$-measurable map $Y_1:\Omega\to R^d$

$$Y_1(\omega)=(\omega)\int_{-\infty}^0 T_{-s}P^+B_0dW(s)-(\omega)\int_0^\infty T_{-s}P^-B_0dW(s). \tag{2.24}$$

Then we have

$$\begin{aligned} Y_1(\theta_t\omega) &= (\theta_t\omega)\int_{-\infty}^0 T_{-s}P^+B_0dW(s)-(\theta_t\omega)\int_0^\infty T_{-s}P^-B_0dW(s) \\ &= (\omega)\int_{-\infty}^t T_{t-s}P^+B_0dW(s)-(\omega)\int_t^\infty T_{t-s}P^-B_0dW(s). \end{aligned}$$

We need to solve the equation

$$Z(t,\omega)=\int_{-\infty}^t T_{t-s}P^+F(Z(s,\omega))+Y_1(\theta_s\omega))ds-\int_t^\infty T_{t-s}P^-F(Z(s,\omega)+Y_1(\theta_s\omega)))ds. \tag{2.25}$$

For this, define

$$C_s^0((-\infty,+\infty),L^2(\Omega))=\{\ f\in C((-\infty,\infty),L^2(\Omega)):f(t,\omega)=f(0,\theta_t\omega) \ \ for\ all \ \ t\in(-\infty,+\infty)\}.$$

We now define for any $z \in C_s^0((-\infty,+\infty),L^2(\Omega))$,

$$\begin{aligned}\mathcal{M}(z)(t,\omega) = &\int_{-\infty}^{t} T_{t-s}P^{+}F(z(s,\omega)+Y_1(\theta_s\omega))ds\\ &-\int_{t}^{+\infty} T_{t-s}P^{-}F(z(s,\omega)+Y_1(\theta_s\omega))ds. \end{aligned} \tag{2.26}$$

It's easy to see that

$$\begin{aligned}\mathcal{M}(z)(0,\theta_t\omega) &= \int_{-\infty}^{0} T_{-s}P^{+}F(z(s,\theta_t\omega)+Y_1(\theta_{s+t}\omega))ds - \int_{0}^{+\infty} T_{-s}P^{-}F(z(s,\theta_t\omega)+Y_1(\theta_{s+t}\omega))ds\\ &= \int_{-\infty}^{0} T_{-s}P^{+}F(z(s+t,\omega)+Y_1(\theta_{s+t}\omega))ds - \int_{0}^{+\infty} T_{-s}P^{-}F(z(s+t,\omega)+Y_1(\theta_{s+t}\omega))ds\\ &= \int_{-\infty}^{t} T_{t-s}P^{+}F(z(s,\omega)+Y_1(\theta_s\omega))ds - \int_{t}^{+\infty} T_{t-s}P^{-}F(z(s,\omega)+Y_1(\theta_s\omega))ds\\ &= \mathcal{M}(z)(t,\omega).\end{aligned}$$

By the similar method in the proof of Proposition 2.4, we can see that $\mathcal{M}$ maps $C_s^0((-\infty,+\infty),L^2(\Omega))$ to itself and $\mathcal{M}(\cdot)(t,\omega)$ is continuous. We can for a fixed $T>0$ and define

$$\begin{aligned}C^0_{T,\alpha}((-\infty,+\infty),\mathcal{D}^{1,2}) := \{&f\in C_s^0((-\infty,+\infty),L^2(\Omega)) : f|_{[0,T)}\in C^0([0,T),\mathcal{D}^{1,2}),\\ &i.e.\ ||f||^2=\sup_{t\in[0,T)}||f(t,\omega)||^2_{1,2}<\infty,\ \text{and for any } t,r\in[0,T)\\ &E|\mathcal{D}_r f(t,\omega)|^2\le 2T^2||\nabla F||^2_\infty\cdot||B_0||^2_\infty\cdot\exp\{2T||\nabla F||^2_\infty|t-r|)\}:=\alpha(t,r),\\ &\sup_{s,r_1,r_2\in[0,T)}\frac{E|\mathcal{D}_{r_1}f(s,\omega)-\mathcal{D}_{r_2}f(s,\omega)|^2}{|r_1-r_2|}<\infty\},\end{aligned}$$

and get $\mathcal{M}$ maps $C^0_{T,\alpha}((-\infty,+\infty),\mathcal{D}^{1,2})$ to itself and $\mathcal{M}(C^0_{T,\alpha}((-\infty,+\infty),\mathcal{D}^{1,2})))|_{[0,T)}$ is relatively compact in $C^0([0,T),L^2(\Omega))$. We need to prove that $\mathcal{M}(C^0_{T,\alpha}((-\infty,+\infty),\mathcal{D}^{1,2})))$ is relatively compact in $C_s^0((-\infty,+\infty),L^2(\Omega))$. Note also for any sequence $\mathcal{M}(z_n)\in\mathcal{M}(C^0_{T,\alpha}((-\infty,+\infty),\mathcal{D}^{1,2})))$, there exists a subsequence, still denoted by $\mathcal{M}(z_n)$ and $Z^*\in C^0([0,T),L^2(\Omega))$ such that

$$E|\mathcal{M}(z_n)(0,\omega)-Z^*(\omega)|^2\to 0,\quad \text{as}\quad n\to\infty.$$

Define

$$Z^*(t,\omega)=Z^*(0,\theta_t\omega).$$

Noting

$$\mathcal{M}(z_n)(0,\theta_\tau\omega)=\mathcal{M}(z_n)(t,\omega),$$

and by the probability preserving of θ, we have

$$\begin{aligned}\sup_{t\in(-\infty,\infty)} E|\mathcal{M}(z_n)(t,\omega)-Z^*(\theta_t\omega)|^2 &= \sup_{t\in(-\infty,\infty)} E|\mathcal{M}(z_n)(0,\theta_t\omega)-Z^*(\theta_t\omega)|^2\\ &= E|\mathcal{M}(z_n)(0,\omega)-Z^*(\omega)|^2\\ &\to 0,\quad \text{as}\quad n\to\infty.\end{aligned}$$

So $\mathcal{M}(C^0_{T,\alpha}((-\infty,+\infty),\mathcal{D}^{1,2})))$ is relatively compact in $C^0_s((-\infty,+\infty),L^2(\Omega))$. Therefore, according to generalized Schauder's fixed point theorem, $\mathcal{M}$ has a fixed point in $C^0_s((-\infty,+\infty),L^2(\Omega))$. That is to say that there exists $Z\in C^0_s((-\infty,+\infty),L^2(\Omega))$ such that for any $t\in(-\infty,+\infty)$, $Z(t,\omega)=Z(0,\theta_t\omega)$ and

$$\begin{aligned}Z(0,\theta_t\omega)=&\int_{-\infty}^{t}T_{t-s}P^{+}F(Z(0,\theta_s\omega)+Y_1(\theta_s\omega))ds\\&-\int_{t}^{+\infty}T_{t-s}P^{-}F(Z(0,\theta_s\omega)+Y_1(\theta_s\omega))ds.\end{aligned}$$

Finally, we add Y_1 defined by the integral equation (2.24) to the above equation and also assume

$$Y(\omega):=Z(0,\omega)+Y_1(\omega).$$

It's easy to see that $Y(\omega)$ satisfies (2.22). ♯

Remark 2.2 *The stochastic periodic solution and stationary point may be non-unique. This is because in the proof of Theorem 2.4 and Theorem 2.6, the generalized Schauder's fixed point argument cannot guarantee the uniqueness of the fixed point. But in fact, the nonuniqueness is the essence of stochastic periodic solutions or the stationary solutions, since (random) dynamical systems may have more than one random periodic solutions or stationary solutions. So one should not expect in general there is only one random periodic solution or stationary solution. Needless to say, it is interesting to study how many random periodic solutions or stationary solutions that a stochastic differential equation can have.*

Remark 2.3 *In Mohammed, Zhang and Zhao [17], they proved the existence of the stationary solution for this semilinear stochastic evolution equation under the conditions that F satisfies the globally bounded and globally Lipschitz conditions and the Lipschitz constant L is with the restriction*

$$L[\mu_{m+1}^{-1}-\mu_m^{-1}]<1.$$

The condition was imposed due to the use of the Banach fixed point theorem and the integral equation is on infinite horizon. It is noted that a similar condition appeared in many literature on random invariant manifolds of random dynamical systems ([1],[7],[8]). Therefore it would be interesting to investigate whether or not the method introduced in this paper can be used to get rid of the critical condition in the invariant manifold theorems.

3 Weaken the Condition of F

Our purpose in this section is to push the results of last section further to find a weaker condition to replace the globally bounded condition for F. Now consider the following equation with a standard cut off function F_N,

$$z_N(t)=\int_{-\infty}^{t}T_{t-s}P^{+}F_N(s,z_N(s)+Y_1(s))ds-\int_{t}^{\infty}T_{t-s}P^{-}F_N(s,z_N(s)+Y_1(s))ds. \tag{3.1}$$

Here the function F_N can be constructed by the standard method

$$F_N(s,y) := F(s, y\frac{|y|\wedge N}{|y|}).$$

It is easy to see that F_N is a function from $(-\infty,\infty)\times R^d \to R^d$ and F_N is bounded no matter whether or not F is bounded. By the previous proof, we have, as F_N is bounded, there exists at least one $z_N(t)$, and the solution depends on N, such that

$$\| z_N \|_\infty \leq \beta_N,$$

where β_N is the radius of a closed ball which depends on N and is dominated by F_N such that

$$\beta_N :=\| F_N \|_\infty (\frac{1}{\mu_{m+1}} - \frac{1}{\mu_m}).$$

The idea here is that if we can prove there exists $\beta' > 0$ which is independent of N, such that for all N,

$$\| z_N \|_\infty \leq \beta'.$$

this is to say we can always choose N big enough to cover every F such that

$$F_N = F,$$

and the globally bounded condition for F will then be possible to be omitted. In the following, we are going to work out the idea. To simplify the notation, we denote $z_N(t)$ by $z(t)$ in (3.1) without any confusion. Consider F satisfies the following condition:

Condition (M)

$$(x,(P^+)F(s,x+a,y+b)) \leq L_1x^2 + L_2y^2 + A_1$$
$$(y,(-P^-)F(s,x+a,y+b)) \leq L_3x^2 + L_4y^2 + B_1$$

where

$$L_1 < \mu_{m+1},\ L_4 < -\mu_m,\ L_2, L_3 \geq 0,\ L_2L_3 < \frac{1}{8}(\alpha+\beta)\beta,$$

and $A_1,\ B_1 \geq 0$ *are constants.*

From (3.1), for $z \in R^d$ and a given $Y_1 \in R^d$, we have

$$z(t) := (z^+(t), z^-(t)),\ Y_1(t) := (Y_1^+(t), Y_1^-(t))$$

where $z^+(t), Y^+(t) \in E^u$ and $z^-(t), Y^-(t) \in E^s$. Then (3.1) can be expressed by two parts

$$z^+(t) = \int_{-\infty}^t T_{t-s}P^+F_N(s, z^+(s)+Y_1^+(s), z^-(s)+Y_1^-(s))ds \tag{3.2}$$

and

$$z^-(t) = -\int_t^\infty T_{t-s}P^-F_N(s, z^+(s)+Y_1^+(s), z^-(s)+Y_1^-(s))ds. \tag{3.3}$$

Under the basis $\{e_i, 1 \leq i \leq d\}$, we assume

$$z_i^-(t) = (z^-(t), e_i),\ \ i = 1, \cdots, m$$
$$z_j^+(t) = (z^+(t), e_j),\ \ j = m+1,\ m+2, \cdots, d$$

Consider the differential forms of (3.2) and (3.3) according to each eigenvalue of A, we have

$$\frac{dz_i^-(t)}{dt} = -\mu_i z_i^-(t) + (F_N(t, z^+(t) + Y_1^+(t), z^-(t) + Y_1^-(t)), e_i),\ \ i = 1, \cdots, m,$$

$$\frac{dz_j^+(t)}{dt} = -\mu_j z_j^+(t) + (F_N(t, z^+(t) + Y_1^+(t), z^-(t) + Y_1^-(t)), e_j),\ \ j = m+1,\ m+2, \cdots, d.$$

For the first m differential equations, we consider the backward integral equations. For the rest inequalities, we consider the forward integral equations. Then we have

$$(z_i^-)^2(t) \leq \int_t^\infty 2\mu_m (z_i^-)^2(s)ds - \int_t^\infty 2z_i^-(s)(F_N(s, z^+(s) + Y_1^+(s), z^-(s) + Y_1^-(s)), e_i)ds,$$
$$(z_j^+)^2(t) \leq \int_{-\infty}^t -2\mu_{m+1}(z_j^+)^2(s)ds + \int_{-\infty}^t 2z_j^+(s)(F_N(s, z^+(s) + Y_1^+(s), z^-(s) + Y_1^-(s)), e_j)ds,$$

where $i = 1, \cdots, m,\ j = m+1,\ m+2, \cdots, d$. Then applying the Gronwall inequality for each differential inequality, we have

$$(z_i^-)^2(t) \leq -2\int_t^\infty e^{-(t-s)2\mu_m} z_i^-(s)(F_N(s, z^+(s) + Y_1^+(s), z^-(s) + Y_1^-(s)), e_i)ds,$$
$$(z_j^+)^2(t) \leq 2\int_{-\infty}^t e^{-(t-s)2\mu_{m+1}} z_j^+(s)(F_N(s, z^+(s) + Y_1^+(s), z^-(s) + Y_1^-(s)), e_j)ds,$$

where $i = 1, \cdots, m,\ j = m+1,\ m+2, \cdots, d$. Now we combine them into two types by writing

$$(z^+)^2(t) = \sum_{j=m+1}^d (z_j^+)^2(t),\ \ (z^-)^2(t) = \sum_{i=1}^m (z_i^-)^2(t)$$

and

$$(z^+(s), P^+F_N(s, z^+(s) + Y_1^+(s), z^-(s) + Y_1^-(s))) = \sum_{j=m+1}^d z_j^+(s)(F_N(s, z^+(s) + Y_1^+(s), z^-(s) + Y_1^-(s)), e_j)$$
$$(z^-(s), P^-F_N(s, z^+(s) + Y_1^+(s), z^-(s) + Y_1^-(s))) = \sum_{i=1}^m z_i^-(s)(F_N(s, z^+(s) + Y_1^+(s), z^-(s) + Y_1^-(s)), e_i)$$

Then (3.2) and (3.3) change to

$$(z^+)^2(t) \leq 2\int_{-\infty}^t e^{-(t-s)2\mu_{m+1}}(z^+(s), P^+F_N(s, z^+(s) + Y_1^+(s), z^-(s) + Y_1^-(s)))ds, \tag{3.4}$$
$$(z^-)^2(t) \leq -2\int_t^\infty e^{-(t-s)2\mu_m}(z^-(s), P^-F_N(s, z^+(s) + Y_1^+(s), z^-(s) + Y_1^-(s)))ds. \tag{3.5}$$

From the inequalities (3.4) and (3.5), we see a hope to weaken the condition for F. This will become a coupling problem. Now we see that function F_N also satisfies the following condition as F satisfies Condition (M):

Condition (M(N))

$$(x, (P^+)F_N(s, x+a, y+b)) \leq L_1 x^2 + L_2 y^2 + A_1$$
$$(y, (-P^-)F_N(s, x+a, y+b)) \leq L_3 x^2 + L_4 y^2 + B_1,$$

where

$$L_1 < \mu_{m+1},\ L_4 < -\mu_m,\ L_2, L_3 \geq 0,$$

and $A_1,\ B_1 \geq 0$ *are constants.*

We notice that $L_1,\ L_2,\ L_3,\ L_4$ and $A_1,\ B_1$ can be chosen to be independent on N since we can deduce this from F. Thus, we have (3.4) and (3.5) change to

$$(z^+)^2(t) \leq 2\int_{-\infty}^{t} e^{-(t-s)2\mu_{m+1}}[L_1(z^+)^2(s) + L_2(z^-)^2(s) + A_1]ds$$

$$(z^-)^2(t) \leq 2\int_{t}^{\infty} e^{-(t-s)2\mu_m}[L_3(z^+)^2(s) + L_4(z^-)^2(s) + B_1]ds.$$

This will lead to

$$(z^+)^2(t) \leq 2\int_{-\infty}^{t} e^{-(t-s)2\mu_{m+1}}[L_1(z^+)^2(s) + L_2(z^-)^2(s)]ds + \frac{A_1}{\mu_{m+1}} \tag{3.6}$$

$$(z^-)^2(t) \leq 2\int_{t}^{\infty} e^{-(t-s)2\mu_m}[L_3(z^+)^2(s) + L_4(z^-)^2(s)]ds - \frac{B_1}{\mu_m}. \tag{3.7}$$

In the next step we will apply the Gronwall inequality and coupling method. This leads to

$$\begin{aligned} e^{t2\mu_{m+1}}(z^+)^2(t) \leq& \int_{-\infty}^{t} e^{s2\mu_{m+1}} 2L_2(z^-)^2(s)ds + \frac{A_1}{\mu_{m+1}} e^{t2\mu_{m+1}} \\ &+ \int_{-\infty}^{t} [e^{s2\mu_{m+1}}(z^+)^2(s)]2L_1 ds. \end{aligned}$$

Then applying the Gronwall inequality to the above inequality, we immediately have

$$e^{t2\mu_{m+1}}(z^+)^2(t) \leq \int_{-\infty}^{t} e^{s2\mu_{m+1}} 2L_2(z^-)^2(s) e^{2L_1(t-s)}ds + \int_{-\infty}^{t} 2A_1 e^{s2\mu_{m+1}} e^{2L_1(t-s)}ds.$$

So it is trivial to see that

$$\begin{aligned} (z^+)^2(t) &\leq \int_{-\infty}^{t} e^{(t-s)2(L_1-\mu_{m+1})} 2L_2(z^-)^2(s)ds + 2A_1\int_{-\infty}^{t} e^{(t-s)2(L_1-\mu_{m+1})}ds \\ &\leq 2\int_{-\infty}^{t} e^{(t-s)2(L_1-\mu_{m+1})} L_2(z^-)^2(s)ds + \frac{A_1}{\mu_{m+1}-L_1}. \end{aligned} \tag{3.8}$$

From (3.7) we have

$$e^{t2\mu_m}(z^-)^2(t) \leq \int_{t}^{\infty} e^{s2\mu_m} 2L_3(z^+)^2(s)ds - \frac{B_1}{\mu_m} e^{t2\mu_m} + \int_{t}^{\infty} [e^{s2\mu_m}(z^-)^2(s)]2L_4 ds.$$

Applying the Gronwall inequality, we have

$$e^{t2\mu_m}(z^-)^2(t) \le \int_t^\infty e^{s2\mu_m} 2L_3 (z^+)^2(s) e^{2L_4(s-t)} ds + \int_t^\infty 2B_1 e^{s2\mu_m} e^{2L_4(s-t)} ds.$$

So it is trivial to see that

$$\begin{aligned}(z^-)^2(t) &\le \int_t^\infty e^{(s-t)2(\mu_m+L_4)} 2L_3 (z^+)^2(s) ds + 2B_1 \int_t^\infty e^{(s-t)2(\mu_m+L_4)} ds \\ &\le 2\int_t^\infty e^{(s-t)2(\mu_m+L_4)} L_3 (z^+)^2(s) ds - \frac{B_1}{\mu_m+L_4}. \end{aligned} \tag{3.9}$$

Observing (3.8) and (3.9), we see that if we prove one of $(z^+)(t)$ and $(z^-)(t)$ is bounded, the other one can be deduced to be bounded automatically. Next, we substitute the term $(z^-)^2(s)$ in (3.8) by the inequality (3.9). Then we can use the change of integration order to get

$$\begin{aligned}(z^+)^2(t) &\le 2\int_{-\infty}^t e^{(t-s)2(L_1-\mu_{m+1})} L_2 \left[2\int_s^\infty e^{(r-s)2(\mu_m+L_4)} L_3 (z^+)^2(r) dr - \frac{B_1}{\mu_m+L_4}\right] ds + \frac{A_1}{\mu_{m+1}-L_1} \\ &\le 4\int_{-\infty}^t e^{(t-s)2(L_1-\mu_{m+1})} L_2 \int_s^\infty e^{(r-s)2(\mu_m+L_4)} L_3 (z^+)^2(r) dr ds + M \\ &= 4L_2L_3 \left[\int_{-\infty}^t \int_{-\infty}^r e^{2(L_1-\mu_{m+1})(t-s)-2(\mu_m+L_4)(s-r)} ds (z^+)^2(r) dr \right. \\ &\qquad \left. + \int_t^\infty \int_{-\infty}^t e^{2(L_1-\mu_{m+1})(t-s)-2(\mu_m+L_4)(s-r)} ds (z^+)^2(r) dr \right] + M \\ &\le \lambda \left[\int_{-\infty}^t e^{2(L_1-\mu_{m+1})(t-s)} (z^+)^2(s) ds + \int_t^\infty e^{2(\mu_m+L_4)(s-t)} (z^+)^2(s) ds \right] + M, \end{aligned} \tag{3.10}$$

where

$$M := \frac{A_1}{\mu_{m+1}-L_1} - \frac{L_2B_1}{(\mu_{m+1}-L_1)(\mu_m+L_4)} > 0,$$

and

$$\lambda := \frac{2L_2L_3}{\mu_{m+1}-L_1-\mu_m-L_4} > 0.$$

Denote

$$\begin{aligned} \alpha &:= \max\{2(\mu_{m+1}-L_1), -2(\mu_m+L_4)\}, \\ \beta &:= \min\{2(\mu_{m+1}-L_1), -2(\mu_m+L_4)\}. \end{aligned}$$

Then $\alpha, \beta > 0$, and

$$(z^+)^2(t) \le M + \lambda \left[\int_{-\infty}^t e^{-\beta(t-s)} (z^+)^2(s) ds + \int_t^\infty e^{-\beta(s-t)} (z^+)^2(s) ds \right].$$

For the above inequality, we consider a variable change for term $\int_t^\infty e^{-\beta(s-t)}(z^+)^2(s)ds$, then

$$\int_t^\infty e^{-\beta(s-t)} (z^+)^2(s) ds = \int_{-\infty}^{-t} e^{-\beta(-s-t)} (z^+)^2(-s) ds.$$

Hence

$$(z^+)^2(t) \le M + \lambda \left[\int_{-\infty}^t e^{-\beta(t-s)} (z^+)^2(s) ds + \int_{-\infty}^{-t} e^{-\beta(-s-t)} (z^+)^2(-s) ds \right]. \tag{3.11}$$

Replacing t by $-t$ into (3.11), we have a new form

$$(z^+)^2(-t) \le M + \lambda \left[\int_{-\infty}^{-t} e^{-\beta(-t-s)} (z^+)^2(s) ds + \int_{-\infty}^{t} e^{-\beta(t-s)} (z^+)^2(-s) ds \right]. \tag{3.12}$$

Adding (3.11) and (3.12) together, we have

$$\begin{aligned}(z^+)^2(t) + (z^+)^2(-t) \le 2M + \lambda \Big[&\int_{-\infty}^{t} e^{-\beta(t-s)} ((z^+)^2(s) + (z^+)^2(-s)) ds \\ &+ \int_{-\infty}^{-t} e^{-\beta(-s-t)} ((z^+)^2(s) + (z^+)^2(-s)) ds \Big].\end{aligned}$$

Observing the above inequality, we find that it becomes an induction problem. Let

$$G(t) = (z^+)^2(t) + (z^+)^2(-t).$$

Then $G(t) \ge 0$ and

$$G(t) \le 2M + \lambda \left[\int_{-\infty}^{t} e^{-\beta(t-s)} G(s) ds + \int_{-\infty}^{-t} e^{-\beta(-s-t)} G(s) ds \right]. \tag{3.13}$$

For the estimation of this inequality, we use the induction method by assuming the starting point $G_1(t) \le 2M$, then

$$\begin{aligned}
G_1(t) &\le 2M \\
G_2(t) &\le 2M + \lambda \int_{-\infty}^{t} e^{-\beta(t-s)} (2M) ds + \lambda \int_{-\infty}^{-t} e^{-\beta(-s-t)} (2M) ds \\
&\le 2M + 2M(\frac{2\lambda}{\beta}) \\
G_3(t) &\le 2M + \lambda \int_{-\infty}^{t} e^{-\beta(t-s)} (2M + 2M(\frac{2\lambda}{\beta})) ds + \lambda \int_{-\infty}^{-t} e^{-\beta(-s-t)} (2M + 2M(\frac{2\lambda}{\beta})) ds \\
&\le 2M + 2M(\frac{2\lambda}{\beta}) + 2M(\frac{2\lambda}{\beta})^2 \\
&\vdots \\
G_m(t) &\le 2M + 2M(\frac{2\lambda}{\beta}) + 2M(\frac{2\lambda}{\beta})^2 + \cdots + 2M(\frac{2\lambda}{\beta})^{m-1} \\
&\vdots
\end{aligned}$$

We see from the induction, if $G_m(t)$ has a uniform bound, we require $\frac{2\lambda}{\beta} < 1$. This means we need

$$\frac{\frac{4L_2L_3}{\mu_{m+1}-L_1-\mu_m-L_4}}{\beta} = \frac{8L_2L_3}{(\alpha+\beta)\beta} < 1.$$

And this leads to

$$L_2L_3 < \frac{1}{8}(\alpha+\beta)\beta.$$

Hence, under this condition, $G_m(t)$ has a uniform bound which does not depend on N. This means $(z^+)^2(t) + (z^+)^2(-t)$ is bounded uniformly in N. And since $(z^+)^2(t)$ and $(z^+)^2(-t)$ must be non-negative, we have $(z^+)^2(t)$ has a uniform bound. Replacing this bound into (3.9), we obtain a bound for $(z^-)^2(t)$. Then we have a bound for $\mid z(t) \mid$, since

$$| z(t) |= ((z^+(t))^2 + (z^-(t))^2)^{\frac{1}{2}}.$$

And this bound completely does not depend on N of F_N. Hence, we can choose N big enough such that $F_N = F$. Then, the globally boundedness condition for F can be omitted now.

To conclude, we have proved the following:

Theorem 3.1 *Assume conditions on A, B_0 in Theorem 2.4. Let $F : (-\infty, \infty) \times R^d \to R^d$ be a continuous map satisfying Condition (M), $\nabla F(t, u)$ be locally bounded, and B_0 and F satisfy Condition (P). Then the semilinear see (1.1) has at least one solution $Y : (-\infty, +\infty) \times \Omega \to R^d$ and solution satisfies*

$$u(t + \tau, t, Y(t, \omega), \omega) = Y(t + \tau, \omega) = Y(t, \theta_\tau \omega) \quad \text{for any} \quad t \geq 0 \qquad a.s. \tag{3.14}$$

Theorem 3.2 *Assume conditions on A, B_0 in Theorem 2.6. Let $F : R^d \to R^d$ be a continuous map satisfying Condition (M) and $\nabla F(u)$ be locally bounded. Then there exists at least one $\mathcal{F}$-measurable map $Y : \Omega \to R^d$ satisfying (2.2).*

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