

On the Noncommutative Residue for Projective Pseudodifferential Operators

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ABSTRACT. A well known result on pseudodifferential operators states that the noncommutative residue (Wodzicki residue) of a pseudodifferential projection vanishes. This statement is non-local and implies the regularity of the eta invariant at zero of Dirac type operators. We prove that in a filtered algebra the value of a projection under any residual trace depends only on the principal part of the projection. This general, purely algebraic statement applied to the algebra of projective pseudodifferential operators implies that the noncommutative residue factors to a map from the twisted K -theory of the co-sphere bundle. We use arguments from twisted K -theory to show that this map vanishes in odd dimensions, thus showing that the noncommutative residue of a projective pseudodifferential projection vanishes. This also gives a very direct proof in the classical setting.

1. Introduction

The algebra of symbols of classical pseudodifferential operators $\Psi\text{DO}_{cl}(X, E)$ on a closed manifold X acting on sections of a vector bundle E can be defined as the quotient of $\Psi\text{DO}_{cl}(X, E)$ by the ideal of smoothing operators. Since pseudodifferential operators are smooth off the diagonal the symbol algebra is localized on the diagonal and it therefore can also be defined locally, using the product expansion formula and the change of charts formula for pseudodifferential operators. That the local heat kernel coefficients and the index of elliptic pseudodifferential operators are locally computable relies on the fact that the index and asymptotic spectral properties of pseudodifferential operators depend only on their class in the symbol algebra. Note that the principal symbol of a pseudodifferential operator is a section of the bundle of endomorphisms of π^*E , where $\pi : T^*X \rightarrow X$ is the canonical projection.

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The bundle of endomorphisms of a complex hermitian vector bundle is a bundle of simple matrix algebras with $*$ -structure. However, not all bundles of simple matrix algebras with $*$ -structure, so-called Azumaya bundles, are isomorphic to endomorphism bundles of hermitian vector bundles. The obstruction is the so-called Dixmier-Douady class in $H^3(X, \mathbb{Z})$. Given an Azumaya bundle \mathcal{A} it is possible to construct algebras of symbols whose principal symbols takes values in the space of sections of the Azumaya bundle $\pi^* \mathcal{A}$ (see for example [MMS05] and the discussion in [MMS06]). Following [MMS05] we refer to such a symbol algebra as the algebra of symbols of projective pseudodifferential operators. For such symbol algebras one can define an index and Mathai, Melrose and Singer [MMS06] proved an index formula for projective pseudodifferential operators, analogous to the Atiyah-Singer index formula. The topological index in this case is a map from twisted K -theory to \mathbb{R} . It has also been shown in [MMS06] that any oriented manifold admits a projective Dirac operator even if the manifold does not admit a spin structure. In this case its index may fail to be an integer.

Another important quantity that depends only on the class of the symbol of a pseudodifferential operator is the so-called Wodzicki residue or noncommutative residue. Up to a factor it is the unique trace on the algebra of pseudodifferential operators. The Wodzicki residue appeared first as a residue of a zeta function measuring spectral asymmetry ([APS76, Wo84]). Wodzicki showed that the regularity of the η -function of a Dirac-type operator at zero – a necessary ingredient to define the η -invariant – follows as a special case from the vanishing of the Wodzicki residue on pseudodifferential projections (as remarked by Brüning and Lesch [BL99] the regularity of the η -function at zero for any Dirac type operator and the vanishing of the Wodzicki residue on pseudodifferential projections are actually equivalent). The regularity of the η -function was proved by Atiyah, Patodi and Singer in [APS76] in the case when X is odd dimensional and later by Gilkey ([Gi81]) in the general case using K -theoretic arguments. Note that whereas the Wodzicki residue can be locally computed its vanishing on pseudodifferential projections is not a local phenomenon. Gilkey [Gi79] constructed a pseudodifferential projection whose residue density is non-vanishing but integrates to zero.

In our paper we show that the Wodzicki residue can also be defined for projective pseudodifferential operators (this has already been observed in [MMS06]) and show that it vanishes on projections in case the dimension of the manifold is odd. Our proof is based on the Leray-Hirsch theorem in twisted K -theory and a purely algebraic result on ‘residue-traces’ in filtered rings.

If L is a filtered ring then we call a linear functional $\tau : L \rightarrow \mathbb{C}$ a residue trace if $\tau(L^{-N}) = \{0\}$ for N large enough. We prove that the value of τ on projections depends only on their class in $L^{(0)} := L/L^{-1}$. Thus, if the map $K_{alg}^0(L) \rightarrow K_{alg}^0(L^{(0)})$ is surjective the map τ descends to a map from the algebraic K -theory of $L^{(0)}$ to

\mathbb{C} . This result can be applied to the Wodzicki residue showing that it descends to a map from twisted K -theory $K^0(S^*X, \pi^*\mathcal{A})$ to \mathbb{C} . We then use the Leray-Hirsch theorem to show that this map actually vanishes. We reduce the problem to positive spectral projections of generalized Dirac operators for which it is known [BG92] that the residue density vanishes.

2. Convolution bundles and Azumaja bundles

Pseudodifferential operators on a smooth closed Riemannian manifold X acting on sections of a vector bundle E can be understood as co-normal distributional sections in the vector bundle $E \boxtimes E^{*1}$ over the space $X \times X$, by identifying the operators with their distributional kernel. The bundle $E \boxtimes E^*$ has the following structure that allows to convolve kernels of integral operators: any element in the fibre over (x, y) may be multiplied by an element in the fibre over (y, z) to give an element in the fibre over (x, z) . Moreover, this multiplication satisfies natural conditions such as associativity. In order to define projective pseudodifferential operators it is convenient to formalize this structure, as we shall do in this section.

2.1. Convolution bundles. Let \mathcal{U} denote an open neighborhood of the diagonal $\Delta(X)$ in $X \times X$ which is symmetric under the reflection map $s : (x, y) \mapsto (y, x)$. Let $p_{ik} : X \times X \times X \rightarrow X \times X$ be defined by $p_{ik}(x_1, x_2, x_3) = (x_i, x_k)$ and set $\tilde{\mathcal{U}} := p_{12}^{-1}(\mathcal{U}) \cap p_{23}^{-1}(\mathcal{U}) \cap p_{13}^{-1}(\mathcal{U})$. Denote by \tilde{p}_{ik} the restriction of the map p_{ik} to $\tilde{\mathcal{U}}$.

DEFINITION 2.1. Let $\pi : F \rightarrow \mathcal{U}$ be a locally trivial vector bundle with typical fibre $\text{Mat}(k)$, the complex $k \times k$ -matrices. We call F a convolution bundle if there exists a homomorphism of vector bundles $m : \tilde{p}_{12}^*F \otimes \tilde{p}_{23}^*F \rightarrow F$ such that the following conditions are satisfied:

- (i) The following diagram is commutative:

$$\begin{array}{ccc} \tilde{p}_{12}^*F \otimes \tilde{p}_{23}^*F & \xrightarrow{m} & F \\ \downarrow & & \downarrow \\ \tilde{\mathcal{U}} & \xrightarrow{\tilde{p}_{13}} & \mathcal{U} \end{array}$$

- (ii) m is associative, i.e., whenever f_{ij} belong to the fibre $F_{(x_i, x_j)}$ then

$$m(m(f_{12} \otimes f_{23}) \otimes f_{34}) = m(f_{12} \otimes m(f_{23} \otimes f_{34})).^2$$

- (iii) There is an atlas $\{\mathcal{O}_\alpha\}$ of \mathcal{U} together with local trivializations

$$\phi_\alpha : \pi^{-1}\mathcal{O}_\alpha \rightarrow \mathcal{O}_\alpha \times \text{Mat}(k),$$

¹ $E \boxtimes E^*$ denotes the external tensor product of E and its dual bundle E^* , i.e., the fibre over a point (x, y) is $E_x \otimes E_y^*$.

²Here we implicitly assume that (x_1, x_2, x_3) , (x_1, x_3, x_4) , and (x_1, x_2, x_4) belong to $\tilde{\mathcal{U}}$.

such that

$$\phi_\alpha(m(f_{12} \otimes f_{23})) = \phi_\alpha(f_{12}) \cdot \phi_\alpha(f_{23})$$

whenever $f_{ij} \in F_{(x_i, x_j)}$ with $(x_1, x_2, x_3) \in \tilde{p}_{12}^{-1}(\mathcal{O}_\alpha) \cap \tilde{p}_{23}^{-1}(\mathcal{O}_\alpha)$.

DEFINITION 2.2. A $*$ -structure on F is a conjugate linear map $*$: $F \rightarrow F$ of vector bundles such that

$$\begin{array}{ccc} F & \xrightarrow{*} & F \\ \downarrow & & \downarrow \\ \mathcal{U} & \xrightarrow{s} & \mathcal{U} \end{array}$$

commutes, such that $(m(f \otimes g))^* = m(g^* \otimes f^*)$, and such that the above local trivializations additionally satisfy

$$\forall f \in \pi^{-1}(\mathcal{O}_\alpha \cap s(\mathcal{O}_\alpha)) : \quad \phi_\alpha(f^*) = \phi_\alpha(f)^*,$$

where the star on the right hand side denotes the hermitian conjugation of matrices. We will refer to a convolution bundle with $*$ -structure as a $*$ -convolution bundle.

Note that $E \boxtimes E^*$ is a particular example for a $*$ -convolution bundle; in this case we can choose $\mathcal{U} = X \times X$. The restriction of a $*$ -convolution bundle F to the diagonal in $X \times X$ is a bundle \mathcal{A} of finite dimensional simple C^* -algebras. Following the literature we refer to such bundles of matrix algebras as Azumaja bundles.

As shown in [MMS05] any Azumaja bundle \mathcal{A} on X gives rise to a convolution bundle near the diagonal in the following way, using an atlas of local trivializations with respect to a good cover³ $\{U_\alpha\}$ of X : The transition functions $\sigma_{\alpha\beta}$ are smooth functions on $U_{\alpha\beta} = U_\alpha \cap U_\beta$ with values in the automorphisms of $\text{Mat}(k)$. Since all automorphisms are inner we can choose local functions $\varphi_{\alpha\beta} : U_{\alpha\beta} \rightarrow SU(k)$ that implement $\sigma_{\alpha\beta}$, i.e., $\sigma_{\alpha\beta}(x)(A) = \varphi_{\alpha\beta}^{-1}(x)A\varphi_{\alpha\beta}(x)$. In general, the functions $\varphi_{\alpha\beta}$ may violate the co-cycle condition and therefore are not the transition functions of a vector bundle. The cocycle condition for the $\sigma_{\alpha\beta}$ together with the condition that the $\varphi_{\alpha\beta}$ are chosen in $SU(k)$ show that any $\varphi_{\alpha\beta}\varphi_{\beta\gamma}\varphi_{\gamma\alpha}$ must be a constant function on $U_\alpha \cap U_\beta \cap U_\gamma$, equal to an k -th root of unity times the identity matrix.⁴ Then we obtain a convolution bundle F with typical fibre $\text{Mat}(k)$ on a neighborhood of the diagonal by choosing the transition functions

$$\phi_{\alpha\beta}(x, y)(A) = \varphi_{\alpha\beta}(x)A\varphi_{\alpha\beta}(y)^{-1}, \quad A \in \text{Mat}(k),$$

on $U_{\alpha\beta} \times U_{\alpha\beta}$. There are also other possible extensions of \mathcal{A} , cf. [MMS06], and Proposition 2.4, below.

³A cover is good if finite intersections of elements therein are either empty or contractible.

⁴On different triple intersections, the resulting unit-root can be different. This induces a torsion element in $H^3(X, \mathbb{Z})$, the Dixmier-Douady class.

REMARK 2.3. *In the sequel it will be occasionally convenient to choose an atlas for F consisting of sets $\mathcal{O}_\alpha := U_\alpha \times U_\alpha$, where $\{U_\alpha\}$ is a good cover of X ; the corresponding trivialisations we shall denote by ϕ_α (so we use the same notation as in Definition 2.1.(iii) above, but possibly have changed the atlas).*

2.2. Transition functions. In the previous section we have seen how an Azumaya bundle leads to a convolution bundle by choosing certain transition functions. Let us now have a closer look to the transition functions of an arbitrary $*$ -convolution bundle. Fix an atlas as explained in Remark 2.3 and let $\phi_{\alpha\beta} : \mathcal{O}_\alpha \cap \mathcal{O}_\beta =: \mathcal{O}_{\alpha\beta} \rightarrow GL(\text{Mat}(k))$ be the transition functions defined by

$$\phi_\beta \circ \phi_\alpha^{-1}((x, y), A) = ((x, y), \phi_{\alpha\beta}(x, y)(A)),$$

Then condition iii) of Definition 2.1 is equivalent to

$$\phi_{\alpha\beta}(x, y)(A)\phi_{\alpha\beta}(y, z)(B) = \phi_{\alpha\beta}(x, z)(AB).$$

In particular,

$$(2.1) \quad (x, x) \mapsto \phi_{\alpha\beta}(x, x) : \mathcal{O}_{\alpha\beta} \cap \Delta(X) \rightarrow \text{Aut}(\text{Mat}(k)).$$

Moreover, Definition 2.2 on the level of the transition functions means that

$$(2.2) \quad \phi_{\alpha\beta}(x, y)(A^*) = \phi_{\alpha\beta}(y, x)(A)^*.$$

PROPOSITION 2.4. *Let F be a $*$ -convolution bundle with transition functions $\phi_{\alpha\beta}$ as described above. Then*

$$(2.3) \quad \phi_{\alpha\beta}(x, y)(A) = \lambda_{\alpha\beta}(x, y)\varphi_{\alpha\beta}(x)A\varphi_{\alpha\beta}(y)^{-1}$$

with mappings

$$\varphi_{\alpha\beta} : \mathcal{O}_{\alpha\beta} \rightarrow SU(k), \quad \lambda_{\alpha\beta} : \mathcal{O}_{\alpha\beta} \rightarrow \mathbb{C},$$

satisfying

$$\lambda_{\alpha\beta}(x, x) = 1, \quad \lambda_{\alpha\beta}(x, y)\lambda_{\alpha\beta}(y, z) = \lambda_{\alpha\beta}(x, z), \quad \lambda_{\alpha\beta}(x, y) = \overline{\lambda_{\alpha\beta}(y, x)},$$

and such that all $\varphi_{\alpha\beta}\varphi_{\beta\gamma}\varphi_{\gamma\alpha}$ are constant functions on their domain of definition, equal to a k -th root of unity times the identity matrix.

PROOF. Combining (2.1) with (2.2) we find $\varphi_{\alpha\beta}$ with

$$\phi_{\alpha\beta}(x, x)(A) = \varphi_{\alpha\beta}(x)A\varphi_{\alpha\beta}(x)^{-1},$$

since all automorphisms of $\text{Mat}(k)$ are inner. Now let us define

$$\phi'_{\alpha\beta}(x, y)(A) = \varphi_{\alpha\beta}(x)^{-1}\phi_{\alpha\beta}(x, y)(A)\varphi_{\alpha\beta}(y).$$

We then have

$$\phi'_{\alpha\beta}(x, x)(A) = A, \quad \phi'_{\alpha\beta}(A)(x, y)\phi'_{\alpha\beta}(y, z)(B) = \phi'_{\alpha\beta}(x, z)(AB).$$

It follows that $\phi'_{\alpha\beta}(x, y)(AB) = \phi'_{\alpha\beta}(x, y)(A)\phi'_{\alpha\beta}(y, y)(B) = \phi'_{\alpha\beta}(x, y)(A)B$ and, analogously, $\phi'_{\alpha\beta}(x, y)(AB) = A\phi'_{\alpha\beta}(x, y)(B)$. Therefore, for all matrices A ,

$$\phi'_{\alpha\beta}(x, y)(\mathbf{1})A = \phi'_{\alpha\beta}(x, y)(A) = A\phi'_{\alpha\beta}(x, y)(\mathbf{1}),$$

where $\mathbf{1}$ is the identity matrix. This shows $\phi'_{\alpha\beta}(x, y)(\mathbf{1})$ is a multiple of the identity matrix. Denoting the corresponding factor by $\lambda_{\alpha\beta}(x, y)$, the claim follows. \square

3. Projective Pseudodifferential Operators

Projective pseudodifferential operators have been defined in [MMS05]. We adapt this definition to fit in our setting of convolution bundles.

3.1. Pseudodifferential operators. To clarify notation let us briefly recall the definition of classical (or polyhomogeneous) pseudodifferential operators on an open subset Ω of \mathbb{R}^n . Let $V \cong \mathbb{C}^k$ be a k -dimensional vector space.

A symbol of order $m \in \mathbb{R}$ is a smooth function $a : \Omega \times \Omega \times \mathbb{R}^n \rightarrow \text{End}(V) = V \otimes V^*$ satisfying estimates

$$\left\| \partial_{\xi}^{\alpha} \partial_{(x,y)}^{\beta} a(x, y, \xi) \right\| \leq C_{\alpha\beta K} (1 + |\xi|)^{m-|\alpha|}$$

for any multi-indices α, β and any compact subset K of $\Omega \times \Omega$, and having an asymptotic expansion $a \sim \sum_{j=0}^{\infty} \chi a_{m-j}$ with a zero-excision function $\chi = \chi(\xi)$ and homogeneous components a_{m-j} , i.e.,

$$a_{m-j}(x, y, t\xi) = t^{m-j} a_{m-j}(x, y, \xi)$$

for all (x, ξ) with $\xi \neq 0$ and all $t > 0$. The pseudodifferential operator $\text{op}(a) : C_0^{\infty}(\Omega, V) \rightarrow C^{\infty}(\Omega, V)$ associated with a is

$$[\text{op}(a)\varphi](x) = \iint e^{i(x-y)\xi} a(x, y, \xi) \varphi(y) dy d\xi, \quad \varphi \in C_0^{\infty}(\Omega, V).$$

An operator $R : C_0^{\infty}(\Omega, V) \rightarrow C^{\infty}(\Omega, V)$ is called smoothing if it has a smooth integral kernel $k \in C^{\infty}(\Omega \times \Omega, \text{End}(V))$, i.e.,

$$(R\varphi)(x) = \int_{\Omega} k(x, y) \varphi(y) dy, \quad \varphi \in C_0^{\infty}(\Omega, V).$$

A pseudodifferential operator of order $m \in \mathbb{R}$ on Ω is an operator of the form $A = \text{op}(a) + R$, where a is a symbol of order m and R is smoothing.

Any pseudodifferential operator $A = \text{op}(a) + R$ of order m can be represented in the form $\text{op}(a_L) + R'$, where $a_L(x, \xi)$ is a y -independent ‘left-symbol’ of order m ; up to order $-\infty$ the left-symbol is uniquely determined by the asymptotic expansion

$$a_L(x, \xi) \sim \sum_{|\alpha|=0}^{\infty} \frac{1}{\alpha!} \partial_{\xi}^{\alpha} D_y^{\alpha} a(x, y, \xi) \Big|_{x=y}.$$

The homogeneous components of A are by definition those of a_L ,

$$\sigma_{m-j}(A)(x, \xi) := (a_L)_{m-j}(x, \xi).$$

By the Schwarz kernel theorem, we can identify A with its distributional kernel

$$K_A \in \mathcal{D}'(\Omega \times \Omega, V \otimes V^*),$$

the topological dual of $C_0^\infty(\Omega \times \Omega; V^* \otimes V)$. It is uniquely defined by the relation

$$\langle K_A, \psi \otimes \varphi \rangle = \langle \psi, A\varphi \rangle, \quad \psi \in C_0^\infty(\Omega, V^*), \quad \varphi \in C_0^\infty(\Omega, V).$$

Denoting by $\text{tr} : V^* \otimes V \rightarrow \mathbb{C}$ the canonical contraction map, we have explicitly

$$\langle K_A, u \rangle = \int_{\Omega} \text{tr}[Au(x, \cdot)](x) dx, \quad u \in C_0^\infty(\Omega \times \Omega; V^* \otimes V).$$

By pseudo-locality, $K_A \in C^\infty(\Omega \times \Omega \setminus \Delta(\Omega), V \otimes V^*)$.

If $U \subset X$ is a coordinate neighborhood, we can pull-back the local operators under the coordinate map. The resulting space of operators we shall denote by $\Psi\text{DO}_{cl}^m(U; \text{End}(V))$, the subspace of smoothing operators by $\Psi\text{DO}^{-\infty}(U; \text{End}(V))$.

3.2. Projective pseudodifferential operators. In the following choose an atlas as explained in Remark 2.3.

DEFINITION 3.1. *Let F be a $*$ -convolution bundle over \mathcal{U} . A distribution $A \in \mathcal{D}'(\mathcal{U}, F)$ is called a projective pseudodifferential operator of order $m \in \mathbb{R}$ if*

- (i) *A is smooth outside the diagonal,*
- (ii) *for any α the distribution $(\phi_\alpha^{-1})^* A|_{U_\alpha \times U_\alpha}$ is the distributional kernel of a pseudodifferential operator $A_\alpha \in \Psi\text{DO}_{cl}^m(U_\alpha; \text{End}(\mathbb{C}^k))$.*

We denote the vector space of m -th order projective pseudodifferential operators by $\Psi\text{DO}_{cl}^m(\mathcal{U}; F)$, the subspace of smoothing elements by $\Psi\text{DO}^{-\infty}(\mathcal{U}; F)$.

The subspace $\text{Diff}^m(\mathcal{U}; F)$ of projective differential operators consists of all projective pseudodifferential operators which are supported on the diagonal.

REMARK 3.2. *If $\mathcal{U} = X \times X$ and $F = E \boxtimes E^*$ for a bundle E over X then $\Psi\text{DO}_{cl}^m(\mathcal{U}; F)$ coincides with $\Psi\text{DO}_{cl}^m(X; E, E)$, the pseudodifferential operators of order m acting on sections into E .*

Though projective pseudodifferential operators, in general, are not operators in the usual sense (i.e., acting between sections of vector bundles) all elements of the standard calculus can be generalized to this setting. In particular, the $*$ -structure gives rise to a conjugation on $\Psi\text{DO}_{cl}^m(\mathcal{U}; F)$, defined by $A^*(x, y) := (A(y, x))^*$ in the distributional sense.

Let A projective pseudodifferential operator with local representatives A_α and A_β , cf. Definition 3.1, where $\mathcal{O}_{\alpha\beta}$ is not empty. By passing to local coordinates on

$U_\alpha \cap U_\beta$, we can associate with A_α and A_β local symbols $a_\alpha(x, \xi)$ and $a_\beta(x, \xi)$, respectively. These symbols are then related by

$$(3.1) \quad \begin{aligned} a_\beta(x, \xi) &= \sum_{|\gamma|=0}^{\infty} \frac{1}{\gamma!} \partial_\xi^\gamma D_y^\gamma \Big|_{y=x} \phi_{\alpha\beta}(x, y) (a_\alpha(x, \xi)) \\ &= \sum_{|\gamma|=0}^{\infty} \frac{1}{\gamma!} \partial_\xi^\gamma D_y^\gamma \Big|_{y=x} [\lambda_{\alpha\beta}(x, y) \varphi_{\alpha\beta}(x) a_\alpha(x, \xi) \varphi_{\alpha\beta}(y)^{-1}], \end{aligned}$$

where the transition function $\phi_{\alpha\beta}$ is as described in (2.2) and Proposition 2.4. Note that this behaviour, in general, differs from the standard case, due the factor $\lambda_{\alpha\beta}(x, y)$. However, (3.1) together with $\lambda_{\alpha\beta}(x, x) = 1$ shows that with A we can associate a well-defined homogeneous principal symbol

$$\sigma_m(A)(x, \xi) \in C^\infty(S^*X, \pi^*\mathcal{A}),$$

where $\pi : S^*X \rightarrow X$ is the canonical co-sphere bundle over X . Vice versa, any given such section can be realized as the principal symbol of a projective pseudodifferential operator.

If the projective pseudodifferential operators A_1 and A_2 are supported in a sufficiently small neighborhood of the diagonal in \mathcal{U} their usual composition

$$(A_1 \circ A_2)(x, z) = \int_X m(A_1(x, y) \otimes A_2(y, z)) dy$$

is a distribution. By passing to local coordinates and using the composition theorems for pseudodifferential operators one can see that $A_1 \circ A_2$ is a projective pseudodifferential operator. The homogeneous principal symbol behaves multiplicative under composition. Of course, any projective pseudodifferential operator can be written as a sum of two operators, where one is smoothing and the other is supported near the diagonal. Summarizing, the coset space

$$(3.2) \quad L_{cl}^*(\mathcal{U}, F) := \Psi DO_{cl}^*(\mathcal{U}, F) / \Psi DO^{-\infty}(\mathcal{U}, F)$$

is a filtered $*$ -algebra. As in the standard case, asymptotic summations of sequences of projective operators of one-step decreasing orders are possible and parametrices (i.e., inverses modulo smoothing remainders) to elliptic elements can be constructed.

THEOREM 3.3. *Let F be a $*$ -convolution bundle and let A be a projective pseudodifferential operator. For $x \in X$ define*

$$\text{WRes}_x(A) := \int_{S_x^*X} \text{tr } a_{-n}(x, \xi) d\sigma(\xi),$$

where $a_{-n}(x, \xi)$, $n = \dim X$, is the homogeneous component of order $-n$ of a symbol of a local representative A_α with $x \in \mathcal{O}_\alpha$, cf. Definition 3.1. Then $\text{WRes}_x(A)$ is well-defined and defines a global density on X . Moreover,

$$\text{WRes}(A) := \int_X \text{WRes}_x(A) dx$$

defines a trace functional on the algebra $L_{cl}^*(\mathcal{U}, F)$, the so-called noncommutative residue or Wodzicki residue.

PROOF. Let A_β be another local representative and $x \in \mathcal{O}_\beta$. Fixing local coordinates on $\mathcal{O}_\alpha \cap \mathcal{O}_\beta$, the local symbols a_α and a_β are related by the asymptotic expansion (3.1). Following the proof in [FGLS96] terms containing a derivative ∂_ξ^γ , $|\gamma| \geq 1$, vanish under integration. We thus obtain the same value for $\text{WRes}_x(A)$ using either $a_\alpha(x, \xi)$ or $a_\beta(x, \xi)$. That $\text{WRes}_x(A)$ transforms as density under changes of coordinates is seen as in the standard case, cf. [FGLS96].

To see that the integral of the residue density defines a trace functional we need to show that it vanishes on commutators $[A, B]$. To this end fix a cover $\{U'_\sigma\}$ of X by coordinate maps together with a subordinate partition of unity, such that $U'_\sigma \cup U'_\rho$ is contained in some U_α whenever $U'_\sigma \cap U'_\rho$ is not empty. We then can write $A = \sum_\alpha A_\sigma$ and $B = \sum_\alpha B_\sigma$ modulo smoothing operators, where the A_σ and B_σ are supported in $\mathcal{O}'_\sigma := U'_\sigma \times U'_\sigma$. Then the commutator $[A, B]$ can be written as a sum of terms $[A_\sigma, B_\rho]$. Such a commutator is smoothing if $\mathcal{O}'_\sigma \cap \mathcal{O}'_\rho$ is empty. Otherwise it is contained in some set \mathcal{O}_α . Therefore the calculation reduces to a local one, which is not different from the one for usual pseudodifferential operators that can be found in [FGLS96]. \square

For purposes below let us establish the following result:

PROPOSITION 3.4. *Let F be a $*$ -convolution bundle and \mathcal{A} be the Azumaya bundle obtained by restricting F to the diagonal. Moreover, let $p \in C^\infty(S^*X, \pi^*\mathcal{A})$ with $p^2 = p$. Then there exists a projective pseudodifferential operator $P \in L_{cl}^0(\mathcal{U}, F)$ which is a projection, i.e., $P^2 = P$, and which has p as its principal symbol. If, additionally, $p^* = p$ then P can be chosen such that $P^* = P$.*

PROOF. The proof follows [Sch01]. Using local coordinates together with a partition of unity we can construct a $P_1 \in L_{cl}^0(\mathcal{U}, F)$ having p as principal symbol. If $V := \mathbb{C} \setminus \{0, 1\}$ then $\lambda - P_1$ is elliptic for any $\lambda \in V$. Thus there is a parametrix $Q(\lambda)$, depending holomorphically on the parameter $\lambda \in V$ (i.e., the local symbols depend holomorphically on λ). Then one takes

$$P = \int_{\mathcal{C}} Q(\lambda) d\lambda,$$

where $\mathcal{C} = \partial U_{1/2}(1)$ is the counter-clockwise oriented boundary of the disc of radius $1/2$ centred in 1 (more precisely, we decompose $Q(\lambda)$ in a sum of local terms and integrate each of these terms separately over \mathcal{C}). If also $p^* = p$ we repeat the above construction, replacing P_1 by P^*P . \square

4. The noncommutative residue in twisted K -theory

4.1. Twisted K -theory. Suppose that \mathcal{A} is an Azumaja bundle over a compact manifold X . The twisted K -theory is defined to be the K -theory of the C^* -algebra of continuous sections $C(X; \mathcal{A})$ of \mathcal{A} .

If $Y \subset X$ is a closed subset then the set of sections $C(X, Y; \mathcal{A})$ vanishing on Y is a closed two-sided ideal in $C(X, Y; \mathcal{A})$ and the quotient by this ideal can be identified with the space of continuous sections $C(Y; \mathcal{A})$ of the Azumaja bundle $\mathcal{A}|_Y$. We therefore have the six term exact sequence as a consequence of the six term exact sequence in the theory of C^* -algebras.

$$\begin{array}{ccccc} K^0(X, Y; \mathcal{A}) & \longrightarrow & K^0(X; \mathcal{A}) & \longrightarrow & K^0(Y; \mathcal{A}) \\ \uparrow & & & & \downarrow \\ K^1(Y; \mathcal{A}) & \longrightarrow & K^1(X; \mathcal{A}) & \longrightarrow & K^1(X, Y; \mathcal{A}) \end{array}$$

where the relative K -groups $K^*(X, Y; \mathcal{A})$ are defined as $K_*(C(X, Y; \mathcal{A}))$.

There is a natural map

$$K_*(C(X, Y; \mathcal{A})) \otimes_{\mathbb{Z}} K_*(C(X)) \mapsto K_*(C(X, Y; \mathcal{A}) \hat{\otimes} C(X)).$$

Here $\hat{\otimes}$ is the tensor product of C^* -algebras which is well defined in this case as $C(X)$ is nuclear. The usual multiplication

$$C(X, Y; \mathcal{A}) \hat{\otimes} C(X) \rightarrow C(X, Y; \mathcal{A})$$

induces a map $K_*(C(X, \mathcal{A}) \hat{\otimes} C(X)) \rightarrow K_*(C(X, \mathcal{A}))$. The composition of these two maps makes $K^*(X, Y; \mathcal{A})$ a module over the \mathbb{Z}_2 -graded ring $K^*(X)$. Choosing $Y = \emptyset$ defines a $K^*(X)$ module structure on $K^*(X; \mathcal{A})$. Note that the morphisms in the six term exact sequence are module homomorphisms.

These observations can be used to prove the following Leray-Hirsch theorem:

THEOREM 4.1. *Let R be a commutative torsion-free ring. Suppose that $\pi : M \xrightarrow{F} X$ is a compact smooth fibre bundle with fibre F over X and let \mathcal{A} be an Azumaja bundle over X . Assume that $K^*(F) \otimes_{\mathbb{Z}} R$ is a free R -module and suppose there exist elements $c_1, \dots, c_N \in K^*(M) \otimes_{\mathbb{Z}} R$ such that the $c_j|_{M_x}$ form a basis for $K^*(M_x) \otimes_{\mathbb{Z}} R$ for every $x \in X$. Then the following map is an isomorphism:*

$$K^*(X; \mathcal{A}) \otimes_{\mathbb{Z}} R^N \longrightarrow K^*(M, \pi^*(\mathcal{A})) \otimes_{\mathbb{Z}} R, \quad (p, \alpha) \mapsto \sum_{j=1}^N \alpha_j \pi^*(p) \cdot c_j.$$

Indeed, the usual proof of the Leray-Hirsch theorem in topological K -theory (see e.g. [H09], Theorem 2.25) can be adapted to our setting in the following way. If

$Y \subset X$ is a closed subset of X , we have the following diagram:

$$\begin{array}{ccccccc} \longrightarrow & K^*(X, Y; \mathcal{A}) \otimes_{\mathbb{Z}} R^N & \longrightarrow & K^*(X; \mathcal{A}) \otimes_{\mathbb{Z}} R^N & \longrightarrow & K^*(Y; \mathcal{A}) \otimes_{\mathbb{Z}} R^N & \longrightarrow \\ & \downarrow \Phi & & \downarrow \Phi & & \downarrow \Phi & \\ \longrightarrow & K^*(\pi^{-1}X, \pi^{-1}Y; \mathcal{A}) \otimes_{\mathbb{Z}} R & \longrightarrow & K^*(\pi^{-1}X; \mathcal{A}) \otimes_{\mathbb{Z}} R & \longrightarrow & K^*(\pi^{-1}Y; \mathcal{A}) \otimes_{\mathbb{Z}} R & \longrightarrow \end{array}$$

Here Φ is defined as in the theorem, $\Phi(p, \alpha) = \sum_{j=1}^N \alpha_j \pi^*(p) \cdot c_j$. The rows of this diagram are exact since tensoring with R^N and R is an exact functor. All maps in the six term exact sequence are natural and therefore the pull back π^* commutes with them. Moreover, the maps in the six term exact sequence for the pair $(\pi^{-1}X, \pi^{-1}Y)$ are $K^*(M)$ module homomorphisms. Thus, the diagram commutes. Since X is a finite cell complex one can proceed in the usual way using the 5-lemma and induction in the number of cells and the dimension to prove the theorem.

4.2. Residue-traces on filtered rings. Let L be a ring with filtration, i.e., $L = L^0 \supset L^{-1} \supset L^{-2} \supset \dots$ with sub-rings L^{-j} and the multiplication induces maps $L^{-i} \times L^{-j} \rightarrow L^{-i-j}$ for any choice of i, j .

A trace functional on L is a map $\tau : L \rightarrow V$ for some vector space V having the following properties:

- (1) τ is linear, $\tau(A + B) = \tau(A) + \tau(B)$ for all $A, B \in L$,
- (2) τ vanishes on commutators, $\tau([A, B]) = \tau(AB - BA) = 0$ for all $A, B \in L$.

We call τ a residue-trace if, additionally,

- (3) there exists a N such that $\tau(A) = 0$ for all $A \in L^{-N}$.

We shall now show that a residue-trace restricted to the set of projections in L is insensitive for lower order terms. The proof is elementary and purely algebraic.

THEOREM 4.2. *Let τ be a residue-trace on L and $P, \tilde{P} \in L$ be two projections, i.e., $P^2 = P$ and $\tilde{P}^2 = \tilde{P}$. If $P - \tilde{P} \in L^{-1}$ then $\tau(P) = \tau(\tilde{P})$.*

PROOF. Set $R = \tilde{P} - P$ and then define

$$A = PRP, \quad B = PR(1 - P), \quad C = (1 - P)RP, \quad D = (1 - P)R(1 - P).$$

Obviously then $\tilde{P} = P + R$ and $R = A + B + C + D$. Using that $P(1 - P) = (1 - P)P = 0$ we obtain

$$\begin{aligned} (P + R)(P + R) &= P + 2A + B + C + A^2 + AB + BC + \\ &\quad + BD + CA + CB + DC + D^2. \end{aligned}$$

On the other hand, using that \tilde{P} is a projection,

$$(P + R)(P + R) = (P + R) = P + A + B + C + D.$$

Equating these two expressions and rearranging of terms yields

$$A^2 + A + BC + D^2 - D + CB + AB + BD + CA + DC = 0.$$

Multiplying this identity from the left and the right with P and $1 - P$, respectively, yields

$$A^2 + A + BC = 0, \quad D^2 - D + CB = 0.$$

Let us rewrite these equations as

$$A(1 + A) = -BC, \quad (-D)(1 + (-D)) = -CB.$$

Multiplying the first equation by $(1 - A)$ yields $A \equiv -BC$ modulo L^{-3} . Multiplying it with $(1 - A + A^2)$ then yields $A \equiv -BC - (BC)^2$ modulo L^{-4} . Proceeding by induction we obtain

$$A \equiv \sum_{k=1}^{\ell} c_{k\ell} (BC)^k \pmod{L^{-(2+\ell)}}$$

for any $\ell \in \mathbb{N}$ with suitable constants $c_{k\ell}$. In the same way, with the same constants $c_{k\ell}$,

$$-D \equiv \sum_{k=1}^{\ell} c_{k\ell} (CB)^k \pmod{L^{-(2+\ell)}}.$$

Therefore we have

$$A + D = \sum_{k=1}^{\ell} c_{k\ell} [B, (CB)^{k-1}C] \pmod{L^{-(2+\ell)}}.$$

Choosing ℓ large enough, we deduce that $\tau(A) + \tau(D) = 0$. Furthermore,

$$\tau(B) = \tau(PR(1 - P)) = \tau((1 - P)PR) = 0$$

and, analogously, $\text{res}(C) = 0$. Altogether we obtain

$$\tau(\tilde{P}) = \tau(P) + \tau(A) + \tau(D) + \tau(B) + \tau(C) = \tau(P)$$

which is the claim we wanted to prove. \square

4.3. The noncommutative residue. We shall show that the noncommutative residue induces a map on twisted K -theory.

PROPOSITION 4.3. *Let \mathcal{A} be the Azumaya bundle obtained by restricting a $*$ -convolution bundle F to the diagonal. The noncommutative residue from Theorem 3.3 descends to a group homomorphism*

$$(4.1) \quad \text{WRes} : K^0(S^*X, \pi^*\mathcal{A}) \rightarrow \mathbb{C},$$

where $\pi : S^*X \rightarrow X$ denotes the co-sphere bundle over X .

PROOF. A typical element in $K^0(S^*X, \pi^*\mathcal{A})$ can be represented by a section $p \in C^\infty(S^*X, \pi^*\text{Mat}_N(\mathcal{A}))$ which is (pointwise) a projection. This is possible, since the natural inclusion of the K -theory of the local C^* -algebra $C^\infty(S^*X, \pi^*\mathcal{A})$ into that of $C(S^*X, \pi^*\mathcal{A})$ is an isomorphism, cf. [B198]. By Proposition 3.4 each such section is the principal symbol of a projective pseudodifferential operator $P \in L^0(X; \text{Mat}_N(F))$ which is a projection. The noncommutative residue of the K -group element is then defined as $\text{WRes}(P)$ in the sense of Theorem 3.3.

We have to show that this map is well-defined. So let $\tilde{p} \in C^\infty(S^*X, \pi^*\text{Mat}_M(\mathcal{A}))$ represent the same element as p does. Let $\tilde{P} \in L^0(X; \text{Mat}_M(F))$ be associated with \tilde{p} . Since p and \tilde{p} are equivalent there exists a unitary $u \in C^\infty(S^*X, \pi^*\text{Mat}_{M+N}(\mathcal{A}))$ such that $u(p \oplus 0_M)u^{-1}$ coincides with $0_N \oplus \tilde{p}$. Let $U \in L^0(X; \text{Mat}_{M+N}(F))$ have u as its principal symbol. Then

$$\text{WRes}(U(P \oplus 0_M)U^{-1}) = \text{WRes}(P \oplus 0_M) = \text{WRes}(P).$$

On the other hand $U(P \oplus 0_M)U^{-1}$ is a projection having the same principal symbol as $0_N \oplus \tilde{P}$. Thus, by Theorem 4.2,

$$\text{WRes}(U(P \oplus 0_M)U^{-1}) = \text{WRes}(0_N \oplus \tilde{P}) = \text{WRes}(\tilde{P}).$$

This shows that the noncommutative residue is independent of the choice of the representative. \square

5. Twisted Dirac operators and connections

Let \mathcal{A} be the Azumaja bundle obtained by restricting a $*$ -convolution bundle F to the diagonal.

DEFINITION 5.1. *A projective connection $\nabla = \nabla^F$ on F is a linear map*

$$Y \mapsto \nabla_Y : C^\infty(X; TX) \longrightarrow \text{Diff}^1(\mathcal{U}; F)$$

satisfying, for any vector field $Y \in C^\infty(X, TX)$ and any function $f \in C^\infty(X)$,

- (1) $\nabla_{fY} = f\nabla_Y$,
- (2) $[\nabla_Y, f] = Yf$ for any $f \in C^\infty(X)$.

It is called a hermitian connection if additionally

$$(3) \nabla_Y^* + \nabla_Y + \text{div } Y = 0$$

(here, f and $\text{div } Y$ are considered as elements of $\text{Diff}^0(\mathcal{U}; F)$).

Note that in case $\mathcal{U} = X \times X$ and $F = E \boxtimes E^*$ for a vector bundle E over X we just recover a usual hermitian connection on E . One can always construct a projective hermitian connection from local hermitian connections by gluing with a partition of unity.

If $\nabla = \nabla^F$ is a projective connection and ϕ_α is a local trivialization of F over $U_\alpha \times U_\alpha$ as described in Remark 2.3, the corresponding local differential operator

$$\nabla_Y^\alpha \in \text{Diff}^m(U_\alpha, \text{End}(\mathbb{C}^k))$$

is of the form

$$\nabla_Y^\alpha = Y + \Gamma_Y^\alpha(x), \quad \Gamma_Y^\alpha \in C^\infty(U_\alpha, \text{Mat}(k)).$$

If we use another trivialisation ϕ_β of F on $U_\beta \times U_\beta$, we have the relation

$$\Gamma_Y^\beta(x) = \phi_{\alpha\beta}(x, x)(\Gamma_Y^\alpha(x)) + Y_y \phi_{\alpha\beta}(x, y)(\mathbf{1})|_{y=x}, \quad x \in U_\alpha \cap U_\beta,$$

where $\mathbf{1}$ is the identity matrix. Thus, analogous to the theory of standard connections, we may describe projective connections by ‘connection matrices’ Γ_Y^α associated to a covering $X = \cup_\alpha U_\alpha$ satisfying the above compatibility relations. For a hermitian connection the connection matrices also have to be skew-symmetric, $\Gamma_Y^\alpha(x)^* = -\Gamma_Y^\alpha(x)$.

Suppose now S is a Clifford module over X and let γ denote the Clifford multiplication. Moreover, let ∇^S be a connection on S which is compatible with the Clifford structure. Writing $\tilde{F} := S \boxtimes S^*$ it is easy to see that $F \otimes \tilde{F}$ is a $*$ -convolution bundle over \mathcal{U} and we can define the projective hermitian connection

$$\nabla := \nabla^F \otimes 1 + 1 \otimes \nabla^S$$

by choosing the corresponding connection matrices as

$$\Gamma_Y^{F,\alpha}(x) \otimes 1 + 1 \otimes \Gamma_Y^{S,\alpha}(x), \quad x \in U_\alpha,$$

where the U_α are chosen in such a way that both F and \tilde{F} are locally trivial over $U_\alpha \times U_\alpha$. Then we can define the twisted Dirac operator

$$D := (1 \otimes \gamma) \circ \nabla \in \text{Diff}^1(\mathcal{U}; F \otimes \tilde{F});$$

in fact, in each local trivialisation ∇ is a usual hermitean connection and we can compose it locally with $1 \otimes \gamma$.

6. Vanishing of the Wodzicki residue

THEOREM 6.1. *If X is an odd dimensional oriented manifold the map WRes of (4.1) vanishes identically.*

As a direct consequence, $\text{WRes}(P) = 0$ for any projection $P \in L^0(\mathcal{U}, F)$.

PROOF OF THEOREM 6.1. Suppose the dimension of X is $n = 2\ell - 1$. Let $S = \bigoplus_{k \text{ even}} \Lambda^k(T^*X)$ denote the bundle of even-degree forms over X and let $*$: $\Lambda^k(T^*X) \rightarrow \Lambda^{n-k}(T^*X)$ the Hodge star operator and denote by d and δ the exterior differential and the co-differential respectively. Define the operator D^S acting on sections of S as

$$D^S = i^\ell * (\delta + (-1)^{k+1} d) \quad \text{on } k\text{-forms.}$$

Then by Proposition 1.22 and 2.8 in [BGV04] this is a generalized Dirac operator, where the Clifford action on S is given by $\gamma(\xi) = i^\ell * (\text{int}(\xi) + (-1)^{k+1} \text{ext}(\xi))$ for $\xi \in T^*X$ and the compatible connection is the Levi-Civita connection. The principal symbol of D^S restricted to the co-sphere bundle is a self-adjoint involution and the projection $\sigma_+(D^S) = \frac{1}{2}(\sigma(D^S) + 1)$ onto its $+1$ eigenspace defines an element in $K^0(S^*X)$. It is well known (see for instance [APS76]) that restriction of this element to each co-sphere S_x^*X equals 2^ℓ times the Bott element on S^{n-1} which together with the class of the trivial line bundle generates $K^0(S^{n-1})$.

For notational convenience denote by $K_{\mathbb{R}}^*(X)$ the groups $K^*(X) \otimes_{\mathbb{Z}} \mathbb{R}$. By Theorem 4.1 applied to the co-sphere bundle of X , any element of $K_{\mathbb{R}}^0(S^*X, \pi^*\mathcal{A})$ can be represented in the form

$$\alpha_0 \pi^*(p) \cdot [\mathbf{1}] + \alpha_1 \pi^*(p) \cdot [\sigma_+(D^S)]$$

for some $\alpha_0, \alpha_1 \in \mathbb{R}$ and some $p \in K^*(X, \mathcal{A})$. Here both the class $[\mathbf{1}]$ of the trivial line bundle and the class $[\sigma_+(D^S)]$ are understood as elements in $K_{\mathbb{R}}^0(S^*X)$. The elements in $\alpha_0 \pi^*(p) \cdot [\mathbf{1}]$ can be represented by projections in $C^\infty(X; \text{Mat}_N(\mathcal{A}))$. Therefore, the noncommutative residue of these elements vanishes. It remains to show that this is also true for the second summand.

To this end let p be a projection in $\text{Mat}_N(C^\infty(X; \mathcal{A}))$. Let us define the new convolution bundle F_p having fibre $p(x)\text{Mat}_N(F)_{(x,y)}p(y) \subset \text{Mat}_N(F)_{(x,y)}$ in (x, y) . We now apply the above construction and build a twisted Dirac operator D_p with respect to $F_p \otimes \tilde{F}$, $\tilde{F} = S \boxtimes S^*$. Then $\sigma_+(D_p)$ represents the class $\pi^*([p]) \cdot [\sigma_+(D^S)]$ in $K^0(S^*X, \pi^*\mathcal{A})$.

The projective differential operator D_p can now be used to construct a certain projection $Q \in L_{cl}^0(X, F_p)$ which has principal symbol as $\sigma_+(D_p)$ on S^*X . In the case of a Dirac type operator D acting on a vector bundle the projection would just be the operator $\frac{1}{2}(|D|^{-1}D + 1)$. The symbol of this projection can be constructed from the a parametrix of D and this construction is local modulo smoothing operators. That is the full symbol of $\frac{1}{2}(|D|^{-1}D + 1)$ modulo smoothing terms in local coordinates depends only on the full symbol of D in these local coordinates. Thus, the construction can be repeated for the operator D_p to yield an element in $L_{cl}^0(X, F_p)$ which we denote by Q or formally $\frac{1}{2}(|D_p|^{-1}D_p + 1)$. By construction $[\sigma(Q)] \in K_{\mathbb{R}}^0(S^*X; \mathcal{A})$ is equal to $\pi^*([p]) \cdot [\sigma_+(D^S)]$.

In [BG92] (Theorem 3.4) Branson and Gilkey have used invariant theory to show that the residue *density* of the positive spectral projection for any generalized Dirac operator vanishes identically. Locally, D_p is a generalized Dirac operator and since the construction of the residue density is local the residue density of Q vanishes as well. So we can conclude that the noncommutative residue of Q vanishes which completes our proof. \square

7. Discussion of even dimensional manifolds

The vanishing of the Wodzicki residue on projective pseudodifferential projections is very likely to hold also in even dimensions. To compute the residue as a map from $K^0(S^*X; \pi^*\mathcal{A})$ it is enough to calculate it on the generators of

$$K_{\mathbb{R}}^0(S^*X; \pi^*\mathcal{A})/\pi^*K_{\mathbb{R}}^0(X; \mathcal{A})$$

which by the Leray-Hirsch theorem is isomorphic to $K_{\mathbb{R}}^1(X; \mathcal{A})$. Of course the result then also holds if $K_{\mathbb{R}}^1(X; \mathcal{A}) = \{0\}$.

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