GAUDIN SUBALGEBRAS AND STABLE RATIONAL CURVES

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Abstract. Gaudin subalgebras are abelian Lie subalgebras of maximal dimension spanned by generators of the Kohno–Drinfeld Lie algebra $\mathfrak{t}_n$. We show that Gaudin subalgebras form a variety isomorphic to the moduli space $\bar{M}_{0,n+1}$ of stable curves of genus zero with $n+1$ marked points. In particular, this gives an embedding of $\bar{M}_{0,n+1}$ in a Grassmannian of $(n-1)$-planes in an $n(n-1)/2$-dimensional space. We show that the sheaf of Gaudin subalgebras over $\bar{M}_{0,n+1}$ is isomorphic to a sheaf of twisted first order differential operators. For each representation of the Kohno–Drinfeld Lie algebra with fixed central character, we obtain a sheaf of commutative algebras whose spectrum is a coisotropic subscheme of a twisted version of the logarithmic cotangent bundle of $\bar{M}_{0,n+1}$.

1. Introduction

The Kohno–Drinfeld Lie algebra $\mathfrak{t}_n$ $(n = 2, 3, \dots)$ over $\mathbb{C}$, see [5, 10], is the quotient of the free Lie algebra on generators $t_{ij} = t_{ji}$, $i \neq j \in \{1, \dots, n\}$ by the ideal generated by the relations

$$\begin{aligned} [t_{ij}, t_{kl}] &= 0, \qquad \text{if } i, j, k, l \text{ are distinct,} \\ [t_{ij}, t_{ik} + t_{jk}] &= 0, \qquad \text{if } i, j, k \text{ are distinct.} \end{aligned}$$

This Lie algebra appears in [10] as the holonomy Lie algebra of the complement of the union of the diagonals $z_i = z_j, i < j$ in $\mathbb{C}^n$. The universal Knizhnik–Zamolodchikov connection [5] takes values in $\mathfrak{t}_n$.

In this paper we consider the abelian Lie subalgebras of maximal dimension contained in the linear span $\mathfrak{t}_n^1$ of the generators t_{ij}. Motivating examples are the algebras considered by Gaudin [6] in the framework of integrable spin chains in quantum statistical mechanics and the Jucys–Murphy subalgebras spanned by $t_{12}, t_{13} + t_{23}, t_{14} + t_{24} + t_{34}, \dots$, appearing in the representation theory of the symmetric group (see [17] and references therein).

Our main result is the classification of Gaudin subalgebras. We show that they are parametrised by the moduli space $\bar{M}_{0,n+1}$ of stable curves of genus zero with $n+1$ marked points (Theorem 2.5). The Gaudin subalgebras parametrised by the open subset $M_{0,n+1}$ are the ones considered originally by Gaudin (with t_{ij} replaced by their image in certain representations of $\mathfrak{t}_n$.) To prove this theorem it is useful to represent $\bar{M}_{0,n+1}$ as a subvariety of a product of projective lines given by explicit equations. We give such a description, proving a variant of a theorem of Gerritzen, Herrlich and van der Put [7], in the Appendix.

Gaudin subalgebras form a locally free sheaf of Lie algebras on $\bar{M}_{0,n+1}$. We describe this sheaf as a sheaf of first order twisted logarithmic differential operators (Theorem 3.3.) For an algebra homomorphism $U\mathfrak{t}_n \to A$ from the universal enveloping algebra of t_n to an associative algebra A, we then get a sheaf of commutative subalgebras $\mathcal{E}_A$ of the $\mathcal{O}_X$-algebra $A \otimes \mathcal{O}_X$ on $X = \bar{M}_{0,n+1}$. We show that its relative spectrum is a coisotropic subscheme of a Poisson variety, a twisted version of the logarithmic cotangent bundle of $\bar{M}_{0,n+1}$ (Corollary 4.3.) For a large class of representations of $U\mathfrak{t}_n$ these spectra, or at least their part over $M_{0,n+1}$, have been recently described in algebro-geometric terms using the Bethe ansatz, see [12, 13] and references therein, and shown to have surprising connection with several other mathematical subjects. It will be interesting to relate these descriptions to the geometry of $\bar{M}_{0,n+1}$. This will be the subject of further investigation.

It is interesting to look at our result in the context of the relation between flag varieties and configuration spaces, whose study was initiated by Atiyah [1–3]. Note that the usual flag variety $U(n)/T^n$ can be naturally viewed as the space of all Cartan subalgebras in the unitary Lie algebra $\mathfrak{u}(n)$. Our result and a parallel between Gaudin and Cartan subalgebras give another link between these two varieties.

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2. Classification of Gaudin subalgebras

Since $\mathfrak{t}_n$ is defined by homogeneous relations, it is graded in positive degrees: $\mathfrak{t}_n = \oplus_{i\geq 1}\mathfrak{t}_n^i$, with $\mathfrak{t}_n^1 = \oplus_{i<j}\mathbb{C}t_{ij}$, $\mathfrak{t}_n^2 = \oplus_{i<j<k}\mathbb{C}[t_{ij}, t_{ik}]$. In particular,

$$\dim(\mathfrak{t}_n^1) = \frac{n(n-1)}{2}, \qquad \dim(\mathfrak{t}_n^2) = \frac{n(n-1)(n-2)}{6}.$$

Definition 2.1. A *Gaudin subalgebra* of $\mathfrak{t}_n$ is an abelian subalgebra of maximal dimension contained in $\mathfrak{t}_n^1$.

We will prove this maximal dimension is $n-1$. It follows from the maximality condition that the central element

$$c_n = \sum_{1\leq i<j\leq n} t_{ij}$$

belongs to all Gaudin subalgebras.

Example 2.2. The *Jucys–Murphy elements* $t_{12}, t_{12}+t_{13}, \dots, \sum_{i=1}^{n-1} t_{in}$ are pairwise commutative and thus span a Gaudin subalgebra. They play an important role in the representation theory of the symmetric group, see Remark 2.6.

Example 2.3. The main class of examples is provided by the spaces [6]

$$G_n(z) = \left\{ \sum_{1\leq i<j\leq n} \frac{a_i - a_j}{z_i - z_j}\, t_{ij}, \quad a \in \mathbb{C}^n \right\}, \tag{1}$$

parametrised by $z \in \Sigma_n/\mathrm{Aff}$, where

$$\Sigma_n = \mathbb{C}^n \smallsetminus \cup_{i<j}\{z \in \mathbb{C}^n \,|\, z_i = z_j\}$$

is the configuration space of n distinct ordered points in the plane and Aff is the group of affine maps $z \mapsto az + b$, $a \neq 0$ acting diagonally on $\mathbb{C}^n$. This parameter space is isomorphic to the moduli space $M_{0,n+1} = ((\mathbb{P}^1)^{n+1} - \cup_{i<j}\{z_i = z_j\})/PSL_2(\mathbb{C})$: the class of z is mapped to the class of $(z_1, \dots, z_n, \infty)$ in $M_{0,n+1}$.

Lemma 2.4. *The dimension of $G_n(z)$ is $n - 1$.*

Proof. The dimension is at most $n - 1$ since there are n parameters $a_1, \dots, a_n$ defined up to a common shift. Taking $a = (1, \dots, 1, 0, \dots, 0)$ with the number of ones ranging from 1 to $n - 1$ we obtain $n - 1$ elements K_j which are linearly independent: $t_{j,j+1}$ appears in K_j with non-vanishing coefficient but not in K_i, $i \neq j$. □

The main result of this section is that Gaudin subalgebras are in one to one correspondence with points in the Knudsen compactification $\bar{M}_{0,n+1}$ of $M_{0,n+1}$, which is a non-singular irreducible projective variety defined over $\mathbb{Z}$ [9]. More precisely, we have the following result.

Theorem 2.5. *Gaudin subalgebras in $\mathfrak{t}_n$ form a nonsingular subvariety of the Grassmannian $G(n-1, n(n-1)/2)$ of $(n-1)$-planes in $\mathfrak{t}_n^1$, isomorphic to $\bar{M}_{0,n+1}$.*

Remark 2.6. To prove this theorem we only use the defining relations of $\mathfrak{t}_n$ and the fact that both the generators t_{ij}, $1 \leq i < j \leq n$ and the brackets $[t_{ij}, t_{ik}]$, $1 \leq i < j < k \leq n$ are linearly independent. Thus our result holds for any quotient of $\mathfrak{t}_n$ with these properties. An important example is the image of $\mathfrak{t}_n$ in the group algebra $\mathbb{C}S_n$ of the symmetric group with commutator bracket, with t_{ij} sent to the transposition of i and j. An approach to the representation theory of S_n based on the simultaneous diagonalization of the image of the Jucys–Murphy elements was proposed in [17]. Another interesting case is given by the homomorphism $\phi\colon \mathfrak{t}_n \mapsto U\mathfrak{o}(n)$ into the universal enveloping algebra of the Lie algebra of the orthogonal group, sending t_{ij} to X_{ij}^2 where X_{ij}, $i < j$ are the standard generators of the Lie algebra $\mathfrak{o}(n)$. The image consists of quantum Hamiltonians of the corresponding Manakov tops [11].

The rest of this section is dedicated to the proof of Theorem 2.5.

Let D_n be the set of all distinct triples (i, j, k) of numbers between 1 and n (i.e., the set of injective maps $\{1, 2, 3\} \to \{1, \dots, n\}$). For $(i, j, k) \in D_n$ denote by $p_{ijk}\colon \mathfrak{t}_n^1 \to \mathbb{C}^3$ the linear map

$$\sum_{i<j} a_{ij} t_{ij} \to (a_{jk}, a_{ik}, a_{ij})$$

where a_{ij} is extended to all pairs by the rule $a_{ji} = a_{ij}$. The map $p\colon D_n \to \mathrm{Hom}(\mathfrak{t}_n^1, \mathbb{C}^3), (i, j, k) \mapsto p_{ijk}$ is equivariant under the natural action of S_3 on D_n and on $\mathbb{C}^3$.

Lemma 2.7. *Let $V \subset \mathfrak{t}_n^1$ be a Gaudin subalgebra. Then, for all $(i, j, k) \in D_n$, $p_{ijk}(V)$ contains $(1, 1, 1)$ and is at most two-dimensional.*

Proof. By the S_3-equivariance it is sufficient to prove the claim for $i < j < k$. The space $p_{ijk}(V)$ contains $p_{ijk}(c_n) = (1, 1, 1)$. Let $a = \sum_{i<j} a_{ij} t_{ij}, b = \sum_{i<j} b_{ij} t_{ij} \in \mathfrak{t}_n^1$. The commutator $[a, b]$ is a linear combination of the linearly independent elements $[t_{ij}, t_{jk}]$, $1 \leq i < j < k \leq n$. Then the equation $[a, b] = 0$ is equivalent to the system

$$a_{ij}b_{jk} - a_{ij}b_{ik} + a_{ik}b_{ij} - a_{ik}b_{jk} + a_{jk}b_{ik} - a_{jk}b_{ij} = 0,$$

$1 \le i < j < k \le n$. These equations are conveniently written in determinant form

$$\det \begin{pmatrix} a_{jk} & b_{jk} & 1 \\ a_{ik} & b_{ik} & 1 \\ a_{ij} & b_{ij} & 1 \end{pmatrix} = 0. \tag{2}$$

Thus $p_{ijk}(V)$ contains at most two linearly independent vectors. □

Thus for each Gaudin subalgebra there exist an S_3 equivariant map $\ell \colon D_n \to (\mathbb{C}^3)^*$ sending (i,j,k) to a linear form ℓ_{ijk} vanishing on $(1,1,1)$ and such that

$$\ell_{ijk} \circ p_{ijk}|_V = 0. \tag{3}$$

If $V = G_n(z)$ eq. (3) is satisfied with the linear forms

$$\ell_{ijk} = (z_j - z_k, z_k - z_i, z_i - z_j).$$

Conversely, we have the following result.

Lemma 2.8. *Let $\ell \colon D_n \to (\mathbb{C}^3)^*$, $(i,j,k) \mapsto \ell_{ijk}$ be an S_3-equivariant map such that $\ell_{ijk}(1,1,1) = 0$ for all (i,j,k). Then*

$$V = \cap_{ijk} \mathrm{Ker}(\ell_{ijk} \circ p_{ijk}).$$

is an abelian Lie subalgebra.

Proof. The vanishing condition implies that $c_n \in V$. If $a, b \in V$ then $p_{ijk}(a), p_{ijk}(b)$ and $(1,1,1)$ belong to a two-dimensional subspace of $\mathbb{C}^3$ and therefore obey (2) for all i,j,k. It follows as in the proof of Lemma 2.7 that $[a,b] = 0$. □

It remains to determine which systems of linear forms ℓ_{ijk} give commuting subspaces of maximal dimension. With respect to the basis t_{ij} of $\mathfrak{t}_n^1$ we can represent the linear forms $\ell_{ijk} \circ p_{ijk}$ as the rows of a matrix L, so that the corresponding commuting subspace is the kernel of L. The matrices arising in this way belong to the following set.

Definition 2.9. Let $n \ge 3$ and $\mathcal{L}_n$ be the space of matrices whose rows are labeled by triples in $D_n^+ = \{(i,j,k), 1 \le i < j < k \le n\}$, whose columns are labeled by pairs in $Z_n^+ = \{(i,j), 1 \le i < j \le n\}$ and such that

1. The matrix elements in the row labeled by $(i,j,k) \in D_n^+$ vanish except possibly those in the columns (j,k), (i,k), (i,j).
2. Each row has at least a non-vanishing matrix element.
3. The sum of the matrix elements in each row is zero.

For example, matrices in $\mathcal{L}_4$ are of the form

$$\begin{array}{c} \\ 123 \\ 124 \\ 134 \\ 234 \end{array} \begin{array}{c} \begin{array}{cccccc} 12 & 13 & 14 & 23 & 24 & 34 \end{array} \\ \begin{pmatrix} c_{123} & b_{123} & 0 & a_{123} & 0 & 0 \\ c_{124} & 0 & b_{124} & 0 & a_{124} & 0 \\ 0 & c_{134} & b_{134} & 0 & 0 & a_{134} \\ 0 & 0 & 0 & c_{234} & b_{234} & a_{234} \end{pmatrix} \end{array}, \tag{4}$$

with nonzero rows and zero row sums.

Proposition 2.10. *Let $L \in \mathcal{L}_n$, $m \le n$. Let L_m be the matrix obtained from L by taking the matrix elements labeled by $D_m^+ \times Z_m^+ \subset D_n^+ \times Z_n^+$. Then $L_m \in \mathcal{L}_m$.*

Proof. The claim is an easy consequence of the definition. □

Lemma 2.11. *Let $L \in \mathcal{L}_n$ and L_m, $3 \leq m \leq n$, the submatrix with labels in $D_m^+ \times Z_m^+$. Then*

$$\operatorname{rank}(L) \geq \frac{(n-1)(n-2)}{2},$$

with equality if and only if

$$\operatorname{rank}(L_m) = \frac{(m-1)(m-2)}{2}, \qquad \text{for all } m = 3, \dots, n.$$

Proof. We claim that there exists a row index set $I = I_3 \cup I_4 \cup \cdots \cup I_n \subset D_n^+$ such that

1. (1) The set I_m has $m-2$ elements; they are of the form (i, j, m) for some $i < j < m$.
2. (2) For each m there are distinct indices $k_1, \dots, k_{m-1} \in \{1, \dots, m\}$ and an ordering $r_1, \dots, r_{m-2}$ of I_m such that the entry of row r_i in column (k_j, m) is zero for $i < j$ and nonzero if $i = j$.

The $(n-1)(n-2)/2$ rows of L labeled by I are then clearly linearly independent, and the same holds for the rows of L_m in $I_3 \cup \cdots \cup I_m$ for all $m \leq n$. It is also clear that if a row of L labeled by $D_m^+ \subset D_n^+$ is a linear combinations of rows labeled by I then it is a linear combinations of rows labeled by I_m. This proves the Lemma assuming the existence of I.

To describe the construction of I it is notationally convenient to think of D_n^+ as the set D_n/S_3 of subsets of three elements and thus to identify $(i, j, k) = (j, i, k) = (i, k, j)$. The row indices I_m can then be taken as $r_i = (\sigma_m(i), \sigma_m(i{+}1), m)$, for some permutation σ_m of $\{1, \dots, m-1\}$ such that the entry of the row r_i in the column $(\sigma_m(i+1), m)$ is nonzero. It remains to prove that such a permutation exists. Let Γ_m be the complete graph with vertex set $\{1, \dots, m\}$. Pick an orientation $i \to j$ on each edge $\{i, j\}$ of Γ_m such that the entry in the row (i, j, m) and the column (j, m) is nonzero. Such an orientation exists since at least two of the entries in columns (i, j), (i, m), (j, m) are nonzero. Then the claim follows from the following simple result of elementary graph theory:

Lemma 2.12. *For any orientation of the edges of a complete graph with k vertices, there exists an oriented path $\sigma(1) \to \sigma(2) \to \cdots \to \sigma(k)$ visiting each vertex exactly once.*

In graph theory such a path is called Hamiltonian and this fact is known as the existence of a Hamiltonian path in any tournament [15].

The proof is by induction: for $k = 1$ there is nothing to prove. Let Γ_k be the complete graph with vertex set $\{1, \dots, k\}$. An orientation of its edges restricts to an orientation of the edges of $\Gamma_{k-1} \subset \Gamma_k$. If we have a path γ on Γ_{k-1} starting at a vertex i and ending at a vertex j then either there is an edge $k \to i$ or $j \to k$ and we can complete γ to a path in Γ_k by adding it, or there exists a step $a \to b$ of γ that can be replaced by $a \to k \to b$ to obtain a path in Γ_k with the required property. □

Corollary 2.13. *Abelian subalgebras lying in $\mathfrak{t}_n^1$ have dimensions at most $n-1$.*

Indeed the rank of a matrix in $\mathcal{L}_n$ is at least $(n-1)(n-2)/2$ and its kernel has dimension at most

$$\frac{n(n-1)}{2} - \frac{(n-1)(n-2)}{2} = n - 1.$$

Lemma 2.14. *Suppose $L \in \mathcal{L}_n$ has minimal rank $(n-1)(n-2)/2$, so that* $\operatorname{Ker} L$ *defines a Gaudin subalgebra. Denote the entries of row (i,j,k) in columns $(i,j),(i,k),(j,k)$ by $a_{ijk}, b_{ijk}, c_{ijk}$ respectively. Then*

$$(a_{ijk} : b_{ijk} : c_{ijk}) = (b_{jik} : a_{jik} : c_{jik}) = (a_{ikj} : c_{ikj} : b_{ikj}) \tag{5}$$

$$a_{ijk} + b_{ijk} + c_{ijk} = 0, \tag{6}$$

$$b_{ijk}c_{ijl}b_{ikl} + c_{ijk}b_{ijl}c_{ikl} = 0, \tag{7}$$

for all $(i,j,k) \in D_n$

Proof. The first two equations are a rephrasing of the S_3-equivariance and the condition $\ell_{ijk}(1,1,1) = 0$. Consider the third equation. By possibly renumbering the vertices we can assume that the four indices are 1,2,3,4. By Lemma 2.11 the submatrix L_4 with labels in $D^+(4) \times Z^+(4)$ has rank 3. This matrix has the form (4). Since the last row is nonzero, the upper left 3×3 minor vanishes. □

We can now conclude the proof of Theorem 2.5. Let Z_n be the subvariety of Gaudin subalgebras in the Grassmannian of $n-1$ planes in $\mathfrak{t}_n^1$. By Lemma 2.7, every Gaudin subalgebra V is contained in $\operatorname{Ker} L$ for some $L \in \mathcal{L}_n$. Since by Lemma 2.8 $\operatorname{Ker} L$ is an abelian subalgebra, it follows by maximality that V is actually equal to $\operatorname{Ker} L$. Let Y_n be the subvariety of $(\mathbb{P}^2)^{D_n}$ defined by the equations (6), (7) for homogeneous coordinates $(a_{ijk} : b_{ijk} : c_{ijk})$. By Lemma 2.14 we then have a map $Z_n \to Y_n$ which is clearly injective. Since the subalgebras $G_n(z)$ are contained in Z_n, Z_n has a component of dimension $n-2$. As we will presently see, Y_n is isomorphic to the non-singular irreducible projective $(n-2)$-dimensional variety $\bar{M}_{0,n+1}$ and therefore $Z_n \to Y_n$ is an isomorphism.

It remains to prove that $Y_n \simeq \bar{M}_{0,n+1}$. In order to do so let us first notice that by (6), Y_n actually lies in $(\mathbb{P}^1)^{D_n}$, where $\mathbb{P}^1 \subset \mathbb{P}^2$ is embedded as $(x : y) \mapsto (x-y : -x : y)$. It is easy to rewrite the remaining equations defining Y_n in these coordinates:

(1) $x_{ikj}x_{ijk} = y_{ikj}y_{ijk}$,
(2) $x_{jik}y_{ijk} = y_{ijk}y_{jik} - x_{ijk}y_{jik}$,
(3) $x_{ijk}y_{ijl}x_{ikl} = y_{ijk}x_{ijl}y_{ikl}$.

It turns out that, by a variant of a Theorem of Gerritzen, Herrlich and van der Put, these are precisely the relations defining $\bar{M}_{0,n+1}$ as a subvariety of $(\mathbb{P}^1)^{D_n}$. We deduce this variant from the original Theorem of Gerritzen, Herrlich and van der Put in the Appendix, see Theorem A.2.

3. The sheaf of Gaudin subalgebras

By Theorem 2.5 Gaudin subalgebras of $\mathfrak{t}_n$ form a family of vector spaces $\mathcal{G}_n$ on $\bar{M}_{0,n+1}$: the fibre at z is the Gaudin subalgebra corresponding to z. The purpose of this section is to identify this family in terms of the geometry of $\bar{M}_{0,n+1}$.

Consider first the Gaudin subalgebras parametrised by $M_{0,n+1}$. Let $\tilde{M}_{0,n+1} = \Sigma_n/\mathbb{C}$, where the group $\mathbb{C} \subset \mathrm{Aff}$ is the translation subgroup. The natural projection $p \colon \tilde{M}_{0,n+1} \to M_{0,n+1}$ is a principal $\mathbb{C}^\times$-bundle. The $\mathfrak{t}_n^1$-valued 1-form

$$\omega = \sum_{i<j} \frac{dz_i - dz_j}{z_i - z_j} t_{ij},$$

is a $\mathbb{C}^\times$-invariant element $\Omega^1(\tilde{M}_{0,n+1}) \otimes \mathfrak{t}_n^1$. The pairing with ω defines a map

$$T_z\tilde{M}_{0,n+1} \to \mathfrak{t}_n^1$$

from the tangent space at $z \in \tilde{M}_{0,n+1}$ to $\mathfrak{t}_n^1$ with image $G_n(z)$, see (1). By Lemma 2.4 this map is injective. Now $G_n(z) = G_n(z')$ if and only if $z' = \lambda z$ for some $\lambda \in \mathbb{C}^\times$. More precisely the action of $\mathbb{C}^\times$ on $\tilde{M}_{0,n+1}$ lifts naturally to $T\tilde{M}_{0,n+1}$ and the invariance of ω implies that ω defines an injective bundle map

$$\begin{array}{ccc} T\tilde{M}_{0,n+1}/\mathbb{C}^\times & \to & M_{0,n+1} \times \mathfrak{t}_n^1 \\ \downarrow & & \downarrow \\ M_{0,n+1} & = & M_{0,n+1} \end{array}$$

Its image is a vector bundle with fibres $G_n(z)$, $z \in M_{0,n+1}$. Moreover $T\tilde{M}_{0,n+1}/\mathbb{C}^\times$ is an extension of the tangent bundle to $M_{0,n+1}$: the kernel of the natural surjective bundle map $T\tilde{M}_{0,n+1}/\mathbb{C}^\times \to TM_{0,n+1}$ is spanned by the class of the Euler vector field

$$E = \sum_{i=1}^n z_i \frac{\partial}{\partial z_i},$$

generating the $\mathbb{C}^\times$-action. Turning to the language of sheaves, more convenient when we pass to $\bar{M}_{0,n+1}$, we thus have an exact sequence of locally free sheaves on $X = M_{0,n+1}$

$$0 \to \mathcal{O}_X \to \mathcal{G}_n \to T_X \to 0. \tag{8}$$

Here T_X is the sheaf of vector fields and $\mathcal{G}_n$ is the sheaf of t_n^1-valued functions whose value at each z lies in $G_n(z)$. For any open set $U \subset M_{0,n+1}$, $\mathcal{G}_n(U)$ may be identified via the map ω with the space of $\mathbb{C}^\times$-invariant vector fields on $p^{-1}(U)$.

As is well-known, invariant vector fields can be identified with first order twisted differential operators on the base manifold:

Lemma 3.1. *Let $p\colon P \to X$ be a principal $\mathbb{C}^\times$-bundle on a smooth variety X, $L = P \times_{\mathbb{C}^\times} \mathbb{C}$ be the associated line bundle where $\mathbb{C}^\times$ acts on $\mathbb{C}$ by multiplication. Then for each open set $U \subset X$, the Lie algebra $T_P(p^{-1}(U))^{\mathbb{C}^\times}$ is isomorphic to the Lie algebra $D^1_{L^\vee}(U)$ of first order differential operators acting on sections of the dual line bundle $L^\vee$ (i.e., twisted by $L^\vee$).*

Proof. A section of $L^\vee$ on U is the same as a function $f\colon p^{-1}(U) \to \mathbb{C}$ such that $f(y \cdot \lambda) = \lambda f(y)$, $y \in p^{-1}(U)$, $\lambda \in \mathbb{C}^\times$. It is clear from this representation that $T_P(p^{-1}(U))^{\mathbb{C}^\times}$ acts on sections of $L^\vee$. Moreover the infinitesimal generator E of the $\mathbb{C}^\times$-action acts by 1. Thus, upon choosing a local trivialization of P, we may write any invariant vector field as $\xi + fE$ where ξ is a vector field on U and $f \in O_X(U)$. This invariant vector field acts on a section as the first order differential operator $\xi + f$. □

As a consequence we have a description of $\mathcal{G}_n$ as a sheaf of twisted first order differential operators:

Proposition 3.2. *Gaudin subalgebras corresponding to points of $M_{0,n+1}$ form a locally free sheaf isomorphic to the sheaf $D^1_{L^\vee}$ of first order differential operators on $M_{0,n+1}$ twisted by the line bundle $L^\vee$ dual to the associated bundle to $\tilde{M}_{0,n+1}$ via the identity character of $\mathbb{C}^\times$.*

Let us now extend this to $\bar{M}_{0,n+1}$.

Let us first recall some known facts about the geometry of $\bar{M}_{0,n+1}$ [7–9]. This space can be defined as the closure in $(\mathbb{P}^1)^{D_n}$ of the image of the injective map

$$\mu\colon M_{0,n+1} = \Sigma_n/\mathrm{Aff} \to (\mathbb{P}^1)^{D_n},$$

sending the class of $z \in \Sigma_n$ to the collection of cross ratios involving the point at infinity

$$\mu_{ijk}(z) = \frac{z_i - z_k}{z_i - z_j} = \frac{(z_i - z_k)(\infty - z_j)}{(z_i - z_j)(\infty - z_k)}, \qquad (i,j,k) \in D_n.$$

Moreover the image is characterized by an explicit set of equations, see Theorem A.2.

The complement of $M_{0,n+1}$ in $\bar{M}_{0,n+1}$ is a normal crossing divisor $D = \cup D_S$, where the union is over all subsets S of $\{1,\dots,n\}$ with at least two and at most $n-1$ elements. The irreducible component D_S is isomorphic to $\bar{M}_{0,m+1} \times \bar{M}_{0,n-m+2}$ and is the closure of the subvariety consisting of stable curves with one nodal point such that the marked points labeled by S are those on the component not containing the point labeled by $n+1$.

Local coordinates on a neighbourhood in $\bar{M}_{0,n+1}$ of a generic point of D_S are the cross ratios $\zeta_r = \mu_{ijr}$, $r \in S \smallsetminus \{i,j\}$, $z_s = \mu_{iks}$, $s \in S^c \smallsetminus \{k\}$, $t = \mu_{ijk}$, for any fixed $i,j \in S$, $k \notin S$ (S^c denotes the complement of S in $\{1,\dots,n\}$). In these coordinates, D_S is given by the equation $t = 0$. The change of variables from these coordinates to the coordinates z_i of $M_{0,n+1}$ is as follows. We may assume that $S = \{1,\dots,m\}$ and choose $i = 1, j = m, k = m+1$ (the general case can be obtained from this by permuting the coordinates). Then for generic $t \neq 0$ the point in $M_{0,n+1}$ with coordinates (ζ, z, t) is

$$[0, t\zeta_2, \dots, t\zeta_{m-1}, t, z_2, \dots, z_{n-m}, 1] \in M_{0,n+1} = \Sigma_n/\mathrm{Aff}. \tag{9}$$

To extend the exact sequence (8) to $\bar{M}_{0,n+1}$ we first show that $\tilde{M}_{0,n+1}$ extends to a principal $\mathbb{C}^\times$-bundle

$$p\colon \tilde{\bar{M}}_{0,n+1} \to \bar{M}_{0,n+1}.$$

This can be seen from the presentation of Theorem A.2. Let H be the kernel of the product map $(\mathbb{C}^\times)^{D_n} \to \mathbb{C}^\times$. Then $P_n = (\mathbb{C}^2 \smallsetminus \{0\})^{D_n}/H$ is a principal $\mathbb{C}^\times$-bundle on $(\mathbb{P}^1)^{D_n}$ and $\tilde{M}_{0,n+1}$ embeds into P_n (via $z \mapsto$ class of $((z_i - z_k, z_i - z_j)_{(i,j,k)\in D_n})$) as the restriction of P_n to the image of $M_{0,n+1}$ in $(\mathbb{P}^1)^{D_n}$. We then define $\tilde{\bar{M}}_{0,n+1}$ to be the restriction of P_n to $\bar{M}_{0,n+1} \subset (\mathbb{P}^1)^{D_n}$.

Recall that the locally free sheaf $T_X\langle -D\rangle$ of logarithmic vector fields on a variety X with a normal crossing divisor D consists of vector fields whose restriction to a generic point of D is tangent to D. It is dual to the sheaf $\Omega^1_X\langle D\rangle$ of logarithmic 1-forms, spanned over $\mathcal{O}_X$ by regular 1-forms and df/f where $f \in \mathcal{O}_X$ with $f \neq 0$ on $X \smallsetminus D$, see [4, Sect. II.3].

Theorem 3.3. *Let L be the associated line bundle $\tilde{\bar{M}}_{0,n+1} \times_{\mathbb{C}^\times} \mathbb{C}$ with the identity character of $\mathbb{C}^\times$. Gaudin subalgebras form a vector bundle on $\bar{M}_{0,n+1}$. As a locally free sheaf, it is isomorphic to the sheaf $D^1_{L^\vee}\langle -D\rangle$ of first order differential operators on $\bar{M}_{0,n+1}$ twisted by $L^\vee$, whose symbol is logarithmic. In particular there is an exact sequence of sheaves on $X = \bar{M}_{0,n+1}$*

$$0 \to \mathcal{O}_X \to \mathcal{G}_n \to T_X\langle -D\rangle \to 0.$$

The embedding of the trivial bundle $\mathcal{O}_X$ sends 1 to $c_n = \sum_{i<j} t_{ij}$.

Proof. Let us introduce the abbreviated notation $X = \bar{M}_{0,n+1}$, $p\colon \tilde{X} = \tilde{\bar{M}}_{0,n+1} \to X$. Let $\tilde{D} = p^{-1}(D)$ be the pull-back to $\tilde{X}$ of the divisor D. Then ω is a form in $\Omega^1(\tilde{X}) \otimes \mathfrak{t}^1_n$ with logarithmic coefficients. Indeed, in the coordinates ζ, z, t of (9) and fibre coordinate $\lambda \in \mathbb{C}^\times$ around a generic point of $\tilde{D}$, ω has the local coordinate expression

$$
\begin{aligned}
\omega \;=\; & \textstyle\sum_{1\le i<j\le m} t_{ij}\, d\log(\zeta_i-\zeta_j) + \sum_{1\le i<j\le m} t_{ij}\, d\log(t) \\
& + \textstyle\sum_{m<i<j\le n} t_{ij}\, d\log(z_i-z_j) + \sum_{1\le i<m<j\le n} t_{ij}\, d\log(z_j) \\
& + c_n\, d\log(\lambda)
\end{aligned}
\tag{10}
$$

with the understanding that $\zeta_0 = 0, \zeta_m = z_n = 1$. Thus ω may be paired with invariant logarithmic vector fields on $\tilde{X}$, which in turn may be identified by Lemma 3.1 with first order differential operators on X, to give a map $D^1_{L^\vee}\langle -D\rangle \to \mathfrak{t}^1 \otimes \mathcal{O}_X$, which is injective on $M_{0,n+1}$. We need to show that the map is injective on all of $X = \bar{M}_{0,n+1}$. Since the locus of non-injectivity is empty or of codimension one, it is sufficient to show that the map is injective as we approach a generic point of the divisor D. This is easy to check using (10).

The embedding of $\mathcal{O}_X$ sends 1 to $\langle \omega, E\rangle = c_n$. □

Thus $\mathcal{G}_n$ is a sheaf of twisted first order differential operators with regular singularities along D, see [16], Sect. 5.2.

Remark 3.4. The divisor class of the line bundle L can be easily computed by choosing a section. The result is

$$[L] = -\sum_{S\supset\{1,2\}} [D_S].$$

4. Coisotropic spectra

Suppose that $U\mathfrak{t}_n \to A$ is a homomorphism of unital algebras, e.g. $A = \mathrm{End}(V)$ for some representation V of $\mathfrak{t}_n$. Then we get a sheaf $\mathcal{E}_A$ of commutative subalgebras of A on $X = \bar{M}_{0,n+1}$ as the image of the symmetric $\mathcal{O}_X$-algebra $S\mathcal{G}_n$. By Theorem 3.3, $\mathcal{G}_n$ is naturally a sheaf of Lie algebras, so that $S\mathcal{G}_n$ is a sheaf of Poisson algebras.

Corollary 4.1. *Let $\varphi\colon U\mathfrak{t}^1_n \to A$ be an algebra homomorphism. Then the kernel of the induced map of sheaves of algebras $\varphi\colon S\mathcal{G}_n \to A\otimes\mathcal{O}_X$, is closed under Poisson brackets.*

Proof. This is basically a consequence of the fact that ω is closed. The map is defined by identifying $\mathcal{G}_n$ with the sheaf of $\mathbb{C}^\times$-invariants of $p_*T_{\tilde{X}}\langle -\tilde{D}\rangle$. It is the algebra homomorphism sending an invariant section ξ to $\omega_A(\xi) = \varphi(\omega(\xi))$. A Poisson bracket of monomials $\xi_1\cdots\xi_k,\ \eta_1,\cdots,\eta_l,\ \xi_i,\eta_j \in \mathcal{G}_n \simeq p_*T_{\tilde{X}}\langle -\tilde{D}\rangle^{\mathbb{C}^\times}$ is sent to

$$
\begin{aligned}
\varphi(\{\xi_1\cdots\xi_k,\eta_1\cdots\eta_l\}) &= \sum_{i,j}\varphi([\xi_i,\eta_j]\xi_1\cdots\hat{\xi}_i\cdots\xi_k\,\eta_1\cdots\hat{\eta}_j\cdots\eta_l)\\
&= \sum_{i,j}\omega_A([\xi_i,\eta_j])\prod_{r\neq i}\omega_A(\xi_r)\prod_{s\neq j}\omega_A(\eta_s)\\
&= \sum_{i,j}\prod_{r<i}\omega_A(\xi_r)\prod_{s<j}\omega_A(\eta_s)\,\omega_A([\xi_i,\eta_j])\prod_{r>i}\omega_A(\xi_r)\prod_{s>j}\omega_A(\eta_s)\\
&= \sum_{i,j}\prod_{r<i}\omega_A(\xi_r)\prod_{s<j}\omega_A(\eta_s)\,(\xi_i\omega_A(\eta_j)-\eta_j\omega_A(\xi_i))\prod_{r>i}\omega_A(\xi_r)\prod_{s>j}\omega_A(\eta_s)\\
&= \sum_{i,j}\prod_{r<i}\omega_A(\xi_r)\prod_{s<j}\omega_A(\eta_s)\,\xi_i\omega_A(\eta_j)\prod_{s>j}\omega_A(\eta_s)\prod_{r>i}\omega_A(\xi_r)\\
&\quad -\sum_{i,j}\prod_{s<j}\omega_A(\eta_s)\prod_{r<i}\omega_A(\xi_r)\,\eta_j\omega_A(\xi_i)\prod_{r>i}\omega_A(\xi_r)\prod_{s>j}\omega_A(\eta_s)\\
&= \sum_{i}\prod_{r<i}\omega_A(\xi_r)\,\xi_i\varphi(\eta_1\cdots\eta_l)\prod_{r>i}\omega_A(\xi_r)-\sum_{j}\prod_{s<j}\omega_A(\eta_s)\eta_j\varphi(\xi_1\cdots\xi_k)\prod_{s>j}\omega_A(\eta_s).
\end{aligned}
$$

In this calculation we use the fact that $\omega_A(\xi_r),\omega_A(\eta_s)$ commute with each other; the peculiar choice of ordering of factors is necessary since their derivatives $\xi_i\omega_A(\eta_j)$, $\eta_j\omega_A(\xi_i)$ do not necessarily commute with them.

It follows that if $\varphi(a)=\varphi(b)=0$ then also $\varphi(\{a,b\})=0$, which is the claim. □

Recall that $\Omega^1_{\tilde{X}}\langle\tilde{D}\rangle$ is locally free and thus the sheaf of sections of a vector bundle, the logarithmic cotangent bundle. Let us denote by $T^*\tilde{X}\langle\tilde{D}\rangle$ the total space of this vector bundle. It is the relative spectrum of the symmetric algebra $ST_{\tilde{X}}\langle-\tilde{D}\rangle$, which is a sheaf of Poisson algebras. Thus $T^*\tilde{X}\langle\tilde{D}\rangle$ is a Poisson variety; the Poisson structure restricts to the usual symplectic structure on the cotangent bundle of $\tilde{M}_{0,n+1}$. The group $\mathbb{C}^\times$ acts on $\tilde{X}$ preserving $\tilde{D}$. This action lifts to a Hamiltonian action on the logarithmic cotangent bundle with moment map $E\in\Gamma(\tilde{X},T_{\tilde{X}}\langle-\tilde{D}\rangle)\subset\Gamma(\tilde{X},ST_{\tilde{X}}\langle-\tilde{D}\rangle)=\mathcal{O}(T^*\tilde{X}\langle\tilde{D}\rangle)$.

Definition 4.2. Let $\alpha\in\mathbb{C}$. The *twisted logarithmic cotangent bundle* $T^*X_\alpha\langle D\rangle$ *with twist* α is the Hamiltonian reduction $E^{-1}(\alpha)/\mathbb{C}^\times$.

By construction $T^*X_\alpha\langle D\rangle$ is a Poisson variety. For $\alpha=0$ it is the logarithmic cotangent bundle of X. By definition, the regular functions on an open set $\tilde{U}=p^{-1}(U)$, $U\subset X$, are section of $(ST_{\tilde{X}}\langle-D\rangle(\tilde{U}))^{\mathbb{C}^\times}/I(\tilde{U})$ where $I(\tilde{U})$ is the ideal generated by $E-\alpha$.

Then Corollary 4.1 can be reformulated as follows.

Corollary 4.3. *Let $U\mathfrak{t}^1_n\to A$ be an algebra homomorphism such that c_n is mapped to $\alpha 1$ and let $\mathcal{E}_A$ be the corresponding sheaf of commutative algebras on $X=\bar{M}_{0,n+1}$. Then the relative spectrum of $\mathcal{E}_A$ is a coisotropic subscheme of the twisted logarithm cotangent bundle $T^*X_\alpha\langle D\rangle$.*

In particular, if A is finite dimensional, then the part of the spectrum over $X^0=M_{0,n+1}$ is a Lagrangian subvariety of the symplectic manifold $T^*X^0_\alpha=(E^{-1}(\alpha)\cap T^*\tilde{X}^0)/\mathbb{C}^\times$.

Appendix A. Stable curves of genus zero

Recall that a stable curve of genus zero with $r \geq 3$ marked points is a pair (C, S) where C is a connected projective algebraic curve of genus 0 whose singularities are ordinary double points and $S = (p_1, \dots, p_r)$ is an ordered set of distinct nonsingular points of C such that each irreducible component of C has at least three special (marked or singular) points. The genus zero condition means that the irreducible components are projective lines whose intersection graph is a tree. The moduli space $\bar{M}_{0,r}$ of stable rational curves with $r \geq 3$ marked point [9] is a smooth algebraic variety of dimension $r-3$ defined over $\mathbb{Q}$. It contains as a dense open set the quotient of the configuration space

$$M_{0,r} = \{z \in (\mathbb{P}^1)^r \mid z_i \neq z_j, \text{for all } i \neq j\}/PSL_2.$$

of r distinct labeled points on the projective line by the diagonal action of $\mathrm{Aut}(\mathbb{P}^1) \simeq PSL_2$.

Here is a simple description of $\bar{M}_{0,r}$, due to Gerritzen, Herrlich and van der Put [7]. For each distinct (i,j,k,l) in $\{1,\dots,r\}$, let $\lambda_{ijkl} \colon M_{0,r} \to \mathbb{P}^1$ be the map sending the class of z to the cross-ratio

$$\lambda_{ijkl}(z) = \frac{(z_i - z_l)(z_j - z_k)}{(z_i - z_k)(z_j - z_l)} \in \mathbb{C} \subset \mathbb{P}^1.$$

Let V_r be the set of distinct quadruples of integers between 1 and r. Then $\bar{M}_{0,r}$ is the closure of the image of the embedding

$$M_{0,r} \to (\mathbb{P}^1)^{V_r}, \quad z \to (\lambda_v(z))_{v \in V_r},$$

sending z to the system of cross-ratios $\lambda_{ijkl}(z) \in \mathbb{P}^1 \smallsetminus \{0, \infty, 1\}$.

The cross-ratios $\lambda_v = \lambda_v(z)$ of a point in $M_{0,r}$ obey the relations

(λ1) $\lambda_{jikl} = 1/\lambda_{ijkl}$ for all distinct i,j,k,l.
(λ2) $\lambda_{jkli} = 1 - \lambda_{ijkl}$ for all distinct i,j,k,l.
(λ3) $\lambda_{ijkl}\lambda_{ijlm} = \lambda_{ijkm}$ for all distinct i,j,k,l,m.

Theorem A.1. *(Gerritzen, Herrlich, van der Put [7]) The subvariety of $(\mathbb{P}^1)^{V_r}$ defined by these relations, more precisely by their version for homogeneous coordinates $\lambda_v = x_v/y_v$:*

(1) $x_{jikl}x_{ijkl} = y_{jikl}y_{ijkl}$ *for all distinct i,j,k,l.*
(2) $x_{jkli}y_{ijkl} = y_{ijkl}y_{jkli} - x_{ijkl}y_{jkli}$ *for all distinct i,j,k,l.*
(3) $x_{ijkl}x_{ijlm}y_{ijkm} = y_{ijkl}y_{ijlm}x_{ijkm}$ *for all distinct i,j,k,l,m,*

is a fine moduli space of stable curves of genus zero. The dense open subvariety $M_{0,r}$ is embedded via the cross-ratios $z \mapsto (x_v : y_v)_{v \in V_r}$, with

$$(x_{ijkl} : y_{ijkl}) = ((z_i - z_l)(z_j - z_k) : (z_i - z_k)(z_j - z_l)).$$

For our purpose it is useful to have a more economical description of the moduli space by taking only cross-ratios involving a distinguished marked point. Let $n = r+1 \geq 2$ and D_n be the set of all distinct triples (i,j,k) of integers between 1 and n.[1] The cross-ratios $\mu_{ijk}(z) = \lambda_{i,n+1,j,k}(z)$ obey

(μ1) $\mu_{ikj} = 1/\mu_{ijk}$, for all distinct i,j,k,
(μ2) $\mu_{jik} = 1 - \mu_{ijk}$, for all distinct i,j,k,

[1] Following [7], we denote the sets of distinct pairs, triples and quadruples by the initials Z, D, V of the corresponding German or Dutch numerals

($\mu 3$) $\mu_{ijk}\mu_{ikl} = \mu_{ijl}$, for all distinct i, j, k, l.

The claim is that the homogeneous version of these relations define $\bar{M}_{0,n+1}$:

Theorem A.2. *The moduli space $\bar{M}_{0,n+1}$ is isomorphic to the subvariety of $(\mathbb{P}^1)^{D_n}$ defined by the equations*

(1) $x_{ikj}x_{ijk} = y_{ikj}y_{ijk}$ *for all* $(i, j, k) \in D_n$.
(2) $x_{jik}y_{ijk} = y_{ijk}y_{jik} - x_{ijk}y_{jik}$, *for all* $(i, j, k) \in D_n$,
(3) $x_{ijk}x_{ikl}y_{ijl} = y_{ijk}y_{ikl}x_{ijl}$ *for all* $(i, j, k, l) \in V_n$.

The open subvariety $M_{0,n+1}$ is embedded via the cross-ratios $z \mapsto (x_d : y_d)_{d\in D_n}$, with

$$(x_{ijk} : y_{ijk}) = ((z_i - z_k)(z_{n+1} - z_j) : (z_i - z_j)(z_{n+1} - z_k))$$

Remark A.3. In [7], $\bar{M}_{0,n+1}$ is considered as a scheme over $\mathbb{Z}$, being defined as a subscheme of $\prod_{v\in V_{n+1}} \mathrm{Proj}(\mathbb{Z}[x_v, y_v])$. Our proof applies also to this more general setting.

Proof. Let us denote by Y_n the subvariety of $(\mathbb{P}^1)^{D_n}$ defined by these relations. We have an obvious map $f\colon \bar{M}_{0,n+1} \to Y_n$, the projection onto the cross-ratios $\mu_{ijk} = (x_{i,n+1,j,k} : y_{i,n+1,j,k})$ with $(i, j, k) \in D_n$. We show that this map is an isomorphism by constructing the inverse map $\mu \mapsto \lambda = g(\mu)$. If one of i, j, k, l is equal to $n + 1$, λ_{ijkl} is obtained using ($\lambda 1$) and ($\lambda 2$) from

$$\lambda_{i,n+1,k,l} = \mu_{ikl} \tag{11}$$

For $(i, j, k, l) \in V_n$, λ_{ijkl} is given by either

$$\lambda_{ijkl} = \frac{\mu_{ikl}}{\mu_{jkl}} \tag{12}$$

or

$$\lambda_{ijkl} = \frac{\mu_{kij}}{\mu_{lij}}, \tag{13}$$

depending on which of the two expressions is defined (i.e. not $0/0$ or ∞/∞.)[2]

We first check that (11)–(13) correctly define a map $Y_n \to (\mathbb{P}^1)^{V_{n+1}}$. First of all ($\lambda 1$) and ($\lambda 2$) say that the map $\lambda\colon V_{n+1} \to \mathbb{P}^1$ is equivariant under the natural action of S_4 on V_{n+1} and on $\mathbb{P}^1$ by fractional linear transformations. Since, by ($\mu 1$) and ($\mu 2$), μ is S_3-equivariant, (11) defines consistently λ_v for v in the S_4-orbit of $(i, n+1, k, l)$. Next, we claim that at least one of the ratios in (12), (13) is defined. Indeed suppose that (12) is not defined because $\mu_{ikl} = \mu_{jkl} = 0$. Then, by ($\mu 1$) and ($\mu 2$), $\mu_{kil} = 1$ and $\mu_{klj} = \mu_{kjl}^{-1} = (1 - \mu_{jkl})^{-1} = 1$. Thus, by ($\mu 3$), $\mu_{kij} = 1$ and (13) is defined. Similarly, if $\mu_{ikl} = \mu_{jkl} = \infty$ then $\mu_{lij} = 1$. The same arguments shows that (12) is defined if (13) is not. It remains to show that if the right-hand sides of both (12), (13) are defined then they are equal. The following identity is useful for this purpose.

Lemma A.4. *Suppose $(\mu_d)_{d\in D_n}$ obey ($\mu 1$), ($\mu 2$) and let $(i, j, k) \in D_n$. Then either $\mu_{ijk}, \mu_{jki}, \mu_{kij}$ are a permutation of $0, \infty, 1$ or none of them belongs to $\{0, \infty, 1\}$. In the latter case their product is -1 or, equivalently,*

$$\mu_{ijk} = -\frac{\mu_{jik}}{\mu_{kij}}.$$

[2]The notation $x_3 = x_1/x_2$ for $x_i = (x_i' : x_i'') \in \mathbb{P}^1$ means $x_3'x_2'x_1'' = x_1'x_2''x_3''$; this defines x_3 given x_1 and x_2 unless x_1 and x_2 are both $0 = (0 : 1)$ or $\infty = (1 : 0)$

Indeed, if $\mu_{ijk} = x$ then $\mu_{jki} = 1/\mu_{jik} = 1/(1-x)$ and $\mu_{kij} = 1 - 1/x$, which implies the Lemma.

This Lemma combined with $(\mu 1), (\mu 3)$ implies the following two identities (holding whenever the expressions are defined):

$$\frac{\mu_{ikl}}{\mu_{jkl}} = \frac{\mu_{ikj}\,\mu_{ijl}}{\mu_{jki}\,\mu_{jil}} = \frac{\mu_{kij}}{\mu_{lij}}, \qquad \frac{\mu_{kij}}{\mu_{lij}} = \frac{\mu_{kil}\,\mu_{klj}}{\mu_{lik}\,\mu_{lkj}} = \frac{\mu_{ikl}}{\mu_{jkl}}. \tag{14}$$

Assume that both right-hand sides of (12),(13) are defined. We have four cases: (a) $\mu_{ikl}, \mu_{jkl} \notin \{0, \infty, 1\}$. Then the second identity in (14) proves that (12),(13) agree. (b) $\mu_{kij}, \mu_{lij} \notin \{0, \infty, 1\}$. Here the first identity implies the claim. (c) $\mu_{ikl} = 0$. Since, by $(\mu 3)$, $\mu_{ikl} = \mu_{ikj}\mu_{ijl}$, we have either $\mu_{ikj} = 0$ or $\mu_{ijl} = 0$. In the first case $\mu_{kij} = 1$ and therefore $\mu_{kjl} = \mu_{kji}\mu_{kil} = 1 \cdot 1 = 1$, implying $\mu_{jkl} = 0$, in contradiction with the assumption that the right-hand side of (12) is defined. In the second case, $\mu_{lij} = 1 - 1/\mu_{ijl} = \infty$ and thus (13) gives $\lambda_{ijkl} = 0$ in agreement with (12). The cases where any of $\mu_{ikl}, \mu_{jkl}, \mu_{kij}, \mu_{lij}$ are 0 or ∞ are treated in the same way. (d) $\mu_{ikl} = 1$. Then $\mu_{kil} = 0$ and thus $\mu_{kij}\mu_{kjl} = 0$. The case $\mu_{kij} = 0$ is covered by (c) so let $\mu_{kjl} = 0$ whence $\mu_{jkl} = 1$. Eq. (12) gives then $\lambda_{ijkl} = 1$. By $(\mu 3)$, $\mu_{ikj} = \mu_{ikl}\mu_{ilj} = 1 \cdot \mu_{ilj}$, $(\mu 1)$ implies $\mu_{kij} = \mu_{lij}$ and thus also (13) gives $\lambda_{ijkl} = 1$. The remaining case $\mu_{kij} = \mu_{lij} = 1$ is treated in the same way by exchanging i, j and k, l.

Thus g is a well-defined morphism $Y_n \to \bar{M}_{0,n+1}$ and by construction $g \circ f$ is the identity.

It remains to show that $\lambda = g(\mu)$ obeys the relations $(\lambda 1)$–$(\lambda 3)$. The first relation $(\lambda 1)$ is obviously satisfied. The relation $(\lambda 2)$ is satisfied by construction if one of i, j, k, l is equal to $n+1$. Also by construction we have

$$\lambda_{ijkl} = \lambda_{klij}, \tag{15}$$

for all distinct i, j, k, l. If $(i, j, k, l) \in V_n$ and $\mu_{jli}, \mu_{kli} \notin \{0, \infty, 1\}$, $(\mu 1)$–$(\mu 3)$ imply

$$\begin{aligned}\lambda_{jkli} = \frac{\mu_{jli}}{\mu_{kli}} = \frac{1-\mu_{lji}}{\mu_{kli}} &= \frac{1-\mu_{ljk}\mu_{lki}}{\mu_{kli}} = \frac{1-(1-\mu_{jlk})\mu_{lki}}{1-\mu_{lki}} \\ &= 1 + \frac{\mu_{jlk}\mu_{lki}}{1-\mu_{lki}} = 1 - \frac{\mu_{ikl}}{\mu_{jkl}} = 1 - \lambda_{ijkl}.\end{aligned}$$

We now consider the degenerate cases. (a) If $\mu_{kli} = 0$ then $\mu_{lki} = 1$ and thus $\mu_{lji} = \mu_{ljk}\mu_{lki} = \mu_{ljk}$ and therefore also $\mu_{jli} = \mu_{jlk}$. Either $\mu_{jli} \neq 0$ so that $\lambda_{jkli} = \infty$ and $\lambda_{ijkl} = \mu_{ikl}/\mu_{jkl} = \infty \cdot \mu_{jlk} = \infty \cdot \mu_{jli} = \infty$ proving the identity; or $\mu_{jli} = 0$ so that the second formula (13) applies and we have

$$\lambda_{jkli} = \frac{\mu_{ljk}}{\mu_{ijk}} = \frac{\mu_{lji}}{\mu_{ijk}} = \frac{1}{\mu_{ijk}} = \mu_{ikj}.$$

On the other hand, since $\mu_{lij} = 1/(1-\mu_{jli}) = 1$,

$$1 - \lambda_{ijkl} = 1 - \frac{\mu_{kij}}{\mu_{lij}} = 1 - \mu_{kij} = \mu_{ikj},$$

proving the claim. (b) If $\mu_{kli} = 1$ then $\mu_{ikl} = 0 = \mu_{lki}$. Then either $\mu_{jkl} = 0$ and (a) with permuted indices gives $\lambda_{ijkl} = 1 - \lambda_{lijk}$ which with (15) implies the claim; or $\mu_{jkl} \neq 0$ and $\lambda_{ijkl} = \mu_{ikl}/\mu_{jkl} = 0$. In this case $\lambda_{jkli} = \mu_{jli}/1 = 1 - \mu_{lji} = 1 - \mu_{ljk}\mu_{lki} = 1 - \mu_{ljk} \cdot 0 = 1$, since $\mu_{ljk} = 1 - 1/\mu_{jkl} \neq \infty$. (c) If $\mu_{kli} = \infty$ then $\mu_{ilk} = 1$ and we are in case (b) up to permutation of i and k, so that we get $\lambda_{jilk} = 1 - \lambda_{kjil}$, which reduces to the claim by using $(\lambda 1)$ and (15). Thus the cases where the denominator μ_{kli} belongs to $\{0,\infty,1\}$ are covered by (a)–(c). (d) The case where the numerator μ_{jli} is in $\{0,\infty,1\}$ is reduced to the previous case by the substitution $i \leftrightarrow k$, $j \leftrightarrow l$. Indeed (a)–(c) give $\lambda_{lijk} = 1 - \lambda_{klij}$, which reduces to ($\lambda$2) by applying (15). This completes the proof of (λ2).

Finally, if μ is generic, (λ3) follows from (μ3):

$$\lambda_{ijkl}\lambda_{ijlm} = \frac{\mu_{ikl}}{\mu_{jkl}}\frac{\mu_{ilm}}{\mu_{jlm}} = \lambda_{ijkm}. \tag{16}$$

This formula applies more generally if one or both λ_{ijkl} and λ_{ijlm} are given by (12): the only tricky case is if the left-hand side of (16) is $0 \cdot \infty$, but in this case there is nothing to prove (see the footnote on page 11.) If both factors are given by (13), we have (trivially):

$$\lambda_{ijkl}\lambda_{ijlm} = \frac{\mu_{kij}}{\mu_{lij}}\frac{\mu_{lij}}{\mu_{mij}} = \lambda_{ijkm}.$$

The remaining case is when for one factor, say λ_{ijkl}, (13) is not defined and for the other, say λ_{ijlm}, (12) is not defined. Then

$$\lambda_{ijkl} = \frac{\mu_{ikl}}{\mu_{jkl}}, \qquad \lambda_{ijlm} = \frac{\mu_{lij}}{\mu_{mij}}.$$

If (13) for λ_{ijkl} is $0/0$, i.e., $\mu_{lij} = 0$ and $\mu_{kij} = 0$, we have that $\mu_{mij} \neq 0$ (since λ_{ijlm} is assumed to be defined by (13)) and $\lambda_{ijlm} = 0/\mu_{mij} = 0$. Also $\lambda_{ijkm} = \mu_{kij}/\mu_{mij}$ is defined and equal to zero and (λ3) is obeyed. Similarly, in the case where (13) is $\lambda_{ijkl} = \infty/\infty$, ($\lambda$3) is obeyed since $\lambda_{ijlm} = \lambda_{ijkm} = \infty$. □

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