ANALYTIC PROPERTIES OF FRACTIONAL SCHRÖDINGER SEMIGROUPS AND GIBBS MEASURES FOR SYMMETRIC STABLE PROCESSES

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Abstract. We establish a Feynman-Kac-type formula to define fractional Schrödinger operators for (fractional) Kato-class potentials as self-adjoint operators. In this functional integral representation symmetric α-stable processes appear instead of Brownian motion. We derive asymptotic decay estimates on the ground state for potentials growing at infinity. We prove intrinsic ultracontractivity of the Feynman-Kac semigroup, introduce the concept of asymptotic intrinsic ultracontractivity, and discuss their relationship and the borderline case of potentials. Finally, we construct Gibbs measures for symmetric stable processes, and prove their uniqueness and support properties.

1. Introduction

The Feynman-Kac formula was originally derived for Schrödinger operators describing the physics of atomic particles. It rests on the basic observation that instead of solving the Schrödinger equation, which is known to be difficult and only possible in a few special cases, one may run a Brownian motion subject to the given potential, and average over the paths in order to get the solution. This probabilistic method proved to be a powerful alternative to the direct operator analysis in studying the properties of the solutions. Feynman-Kac-type formulae were subsequently extended to cover more complex models of quantum field theory and solid state physics (see a systematic discussion in [47]). Due to the presence of the Laplace operator, however, random processes with continuous paths remained a key object in these functional integral representations.

In the recent paper [33] generalized Schrödinger operators of the form

$$H = \Psi(-\Delta) + V \tag{1.1}$$

have been introduced, where Ψ is a so called Bernstein function. An example to this class are the fractional Schrödinger operators

$$H_\alpha = (-\Delta)^{\alpha/2} + V, \quad 0 < \alpha < 2. \tag{1.2}$$

These pseudo-differential operators are non-local, unlike usual Schrödinger operators (obtained for $\Psi(x) = x$), having a range of specific properties, appearing in various models of current interest in mathematical physics, financial mathematics and other areas of applied mathematics. Due to the fact that Bernstein functions with vanishing right limits at the origin are in a one-to-one correspondence with subordinators, the operators H generate subordinate Brownian motion. These are non-Gaussian Lévy processes with càdlàg paths (i.e., right continuous paths with left limits) having jump discontinuities. For instance, fractional Schrödinger operators generate symmetric α-stable processes, and a Feynman-Kac-type formula holds in which this process replaces Brownian motion.

Our main goal in this paper is to derive spectral and analytic properties of semigroups generated by fractional Schrödinger operators with a large class of potentials by developing methods

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of functional integration. Work in this direction was initiated by the paper [17] considering the operator

$$H = \sqrt{-\Delta + m^2} - m + V, \quad m > 0, \tag{1.3}$$

describing a semi-relativistic quantum particle. In fact, this case corresponds to the specific Bernstein function $\Psi(u) = \sqrt{u + m^2} - m$ for which explicit formulae for the probability distribution etc of the generated process generated are available.

In the case of H_α explicit distributional formulae for the related α-stable processes are known only in three cases, for $\alpha = 1/2$ (Lévy distribution), $\alpha = 1$ (Cauchy distribution) and $\alpha = 2$ (Gaussian distribution), therefore sufficiently powerful tools are needed in order to cover the full range of $\alpha \in (0, 2)$. The present paper is our first step towards establishing some basic properties of generalized Schrödinger semigroups by considering semigroups generated by H_α. In a forthcoming work we address the case of operators of the type $H_{\alpha,m} = (-\Delta - m^{2/\alpha})^{\alpha/2} - m + V$, $m > 0$, and, more generally, the case of Bernstein functions of the Laplacian. Building on these results, one of the applications we are interested in is to add further operators and study ground state properties of Hamiltonians describing (semi)relativistic quantum field models. (For an application of functional integration methods to non-relativistic quantum field theory see [7, 45, 46, 34, 47].)

Here is a summary of the main results of this paper.

(**1**) First we define fractional Schrödinger operators H_α for bounded potentials V as self-adjoint operators and derive a Feynman-Kac-type formula involving symmetric α-stable processes (Theorem 3.1), i.e.,

$$\left(e^{-tH_\alpha} f\right)(x) = \mathbf{E}^x\left[e^{-\int_0^t V(X_s)ds} f(X_t)\right], \quad t > 0$$

where $(X_t)_{t\geq 0}$ is a symmetric α-stable process and the expectation at the right hand side is taken with respect to its measure. Then we define a large class of potentials, which we call fractional Kato-class (Definition 3.1). In general, the potentials we consider can be decomposed into a locally Kato-class positive part (which has a killing effect on the paths of the process) and a Kato-class negative part (which has a mass generating effect), while they also can have strong local singularities. We furthermore show that the right hand side in the above formula makes sense when extended to Kato-class and gives rise to a strongly continuous symmetric one-parameter (Feynman-Kac) semigroup (Theorem 3.2). By this procedure we are able to define fractional Schrödinger operators for Kato-decomposable potentials by identifying them as the generators of the so obtained Feynman-Kac semigroups.

(**2**) Next we consider pinning Kato-decomposable potentials $V(x) \to \infty$ as $|x| \to \infty$ for which the Feynman-Kac semigroup is compact, and derive ground state estimates. We obtain that, roughly, the ground state (eigenfunction at the bottom of the spectrum) φ_0 decays like

$$\varphi_0(x) \sim |x|^{-d-\alpha}, \quad |x| \to \infty.$$

This slow decay is in sharp contrast with the (super)exponential fall-off of ground states for pinning potentials known in the case of Schrödinger operators [15], or the case in (1.3) where exponential decay is obtained for $m \neq 0$. This behaviour is due to the heavy tails of stable processes, which is a common feature of non-Gaussian Lévy processes. We furthermore establish lower and upper bounds of the ground state for all values of x (Theorem 4.1 and corollaries) which differ by a correction factor from the above asymptotics, in particular, the correction is proportional to $1/|V(x)|$ for potentials comparable outside a compact set.

(**3**) We prove basic regularity properties of the kernel of the Feynman-Kac semigroup (Theorem 3.3), and address intrinsic ultracontractivity (IUC) of the semigroup in detail. In particular, we offer a characterization of IUC for pinning Kato-decomposable potentials (Theorem 5.1) and show that the borderline case is given, roughly, by potentials growing faster than logarithmically (Theorem 5.2, Corollary 5.2). This also contrasts the case of Schrödinger semigroups where the classic result [24] shows that IUC is obtained for potentials growing faster than quadratically. We offer a heuristic explanation of this qualitative difference by pointing out that while Brownian motion typically has to fluctuate very much to escape from regions where the potential causes a high killing rate, processes with discontinuous paths can typically escape through a jump. We also propose the concept of asymptotically intrinsic ultracontractivity (AIUC), which is a weaker property than IUC, and discuss AIUC of fractional Schrödinger operators.

(**4**) The kernel of the Feynman-Kac semigroup can be expressed as

$$u(t-s,x,y) = \int_\Omega e^{-\int_s^t V(X_r(\omega))dr} d\nu^{x,y}_{[s,t)}(\omega),$$

where $\nu^{x,y}_{[s,t)}$ is the measure of the stable bridge process starting from $x \in \mathbf{R}^d$ at time s and ending in $y \in \mathbf{R}^d$ at time t, and $\Omega = D_{\rm r}(\mathbf{R}, \mathbf{R}^d)$ is the space of two-sided càdlàg paths. By considering the exponential factor $e^{-\int_s^t V(X_r)dr}$ as a density with respect to the bridge measure, upon normalization the right hand side of the Feynman-Kac formula above gives rise to a conditional probability measure $\mu_T(E|X_s = x, X_t = y) = \mu_T^{x,y}(E)$ having the structure of a Gibbs measure indexed by the bounded intervals of $\mathbf{R}$. More generally, we consider two-sided stable processes and define the integrals in the exponent over intervals $[-T,T] \subset \mathbf{R}$. Heuristically, the so obtained Gibbs measure describes the probability of a path running over the time interval $[-T,T]$ given the pin-down conditions at the two ends of the interval. Due to the applications we have in mind we are interested in the existence of Gibbs measures over the whole time line. Since for a typical choice of Kato-decomposable potential $\int_{\mathbf{R}} V(X_r(\omega))dr$ may be almost surely infinite, Gibbs measures for the two-sided process on $\mathbf{R}$ cannot be defined directly. Therefore we say that a Gibbs measure on the line is a probability measure whose family of conditional probabilities with respect to all pin-down conditions (or equivalently, boundary paths) has a version coinciding with the given family $(\mu_T^{x,y})_{T>0}$, i.e., whether there is a probability measure μ on the space of càdlàg paths such that

$$\mu(E) = \int_E \mu_T(E|X_s = x, X_t = y)\mu(dX), \quad \forall T > 0,\, x,y \in \mathbb{R}^d$$

for all cylinder sets E and the family of Borel σ-fields $\mathcal{F}_{[-T,T]^c}$ generated by the coordinate process outside $[-T,T]$. We address existence, uniqueness and support properties of such Gibbs measures for Kato-decomposable potentials. In particular, we show that if the Feynman-Kac semigroup is AIUC, then the Gibbs measure is unique and supported on the full path space (Theorem 6.1), while for other potentials we identify a full measure subset of càdlàg paths on which the unique Gibbs measure is supported (Theorem 6.2). The properties of such Gibbs measures can then be used in a powerful way in the study of spectral and ground state properties of quantum field models (see references above).

The paper is organized as follows. Section 2 contains some preparatory material. We show how α-stable processes can be realized in terms of subordinate Brownian motion, introduce two-sided stable processes, and discuss basic definitions and facts on the potential theory of stable processes and stable bridges. In Section 3 we establish the Feynman-Kac-type formula relating fractional Schrödinger operators with symmetric stable processes, introduce fractional Kato-class, and derive some results on potential theory of these operators needed later. In Section 4 we derive ground state estimates for Kato-decomposable potentials for which the Feynman-Kac semigroup is compact. Section 5 is devoted to discussing intrinsic ultracontractivity and asymptotic intrinsic ultracontractivity. In Section 6 we construct Gibbs measures for symmetric stable processes and discuss their uniqueness and support properties.

2. **Preliminaries**

2.1. **Symmetric α-stable processes as subordinate Brownian motion**

Let $(\Omega_X, \mathcal{F}_X, \mathbf{P}_X)$ be a probability space and $(X_t)_{t\geq 0}$ an $\mathbf{R}^d$-valued symmetric α-stable process on it, with $d \geq 1$ and $\alpha \in (0,2)$. $(X_t)_{t\geq 0}$ is a Lévy process, in particular it has independent and stationary increments. In this paper we are interested in the case of non-Gaussian stable processes only, therefore do not include the case $\alpha = 2$. We use the notations $\mathbf{P}^x$ and $\mathbf{E}^x$, respectively, for the distribution and the expected value of the process starting in $x \in \mathbf{R}^d$ at time $t = 0$; for simplicity we do not indicate the measure in subscript (while we do when have any other measure or process). The characteristic function of $(X_t)_{t\geq 0}$ is given by

$$\mathbf{E}^0[e^{i\xi X_t}] = e^{-t|\xi|^\alpha}, \quad \xi \in \mathbf{R}^d,\, t \geq 0. \tag{2.1}$$

Denote $[0,\infty) = \mathbf{R}^+$. As a Lévy process, $(X_t)_{t\geq 0}$ has a version with paths in $D_{\rm r}(\mathbf{R}^+;\mathbf{R}^d)$, i.e., the space of right continuous functions $\mathbf{R}^+ \to \mathbf{R}^d$ with existing left limits (i.e., càdlàg functions) and in $D_{\rm l}(\mathbf{R}^+;\mathbf{R}^d)$, i.e., the space of left continuous functions $\mathbf{R}^+ \to \mathbf{R}^d$ with existing right limits (i.e., càglàd functions).

Recall that a subordinator $(S_t)_{t\geq 0}$ on a given probability space $(\Omega_S, \mathcal{F}_S, \mathbf{P}_S)$ is an almost surely non-decreasing $\mathbf{R}^+$-valued Lévy process starting at 0. An example is the $(\alpha/2)$-stable subordinator $(S_t)_{t\geq 0}$ uniquely determined by its Laplace transform

$$\mathbf{E}^0_{\mathbf{P}_S}[e^{-\lambda S_t}] = e^{-t\lambda^{\alpha/2}}, \quad t \geq 0,\, \lambda \geq 0. \tag{2.2}$$

We will use the notation $(S_t)_{t\geq 0}$ for this specific subordinator below.

Consider standard Brownian motion $(B_t)_{t\geq 0}$ on a given probability space $(\Omega_W, \mathcal{F}_W, \mathbf{P}_W)$, where $\mathbf{P}_W$ is Wiener measure. Clearly,

$$\mathbf{E}^0_{\mathbf{P}_W}[e^{i\xi B_t}] = e^{-t|\xi|^2}, \quad \xi \in \mathbf{R}^d,\, t \geq 0. \tag{2.3}$$

It is a standard fact [4, 8] that any symmetric α-stable process $(X_t)_{t\geq 0}$ can be obtained as a random time change of Brownian motion where this random time process is an $(\alpha/2)$-stable subordinator $(S_t)_{t\geq 0}$. It is convenient to consider the processes $(B_t)_{t\geq 0}$ and $(S_t)_{t\geq 0}$ on two different probability spaces $(\Omega_W, \mathcal{F}_W, \mathbf{P}_W)$ and $(\Omega_S, \mathcal{F}_S, \mathbf{P}_S)$. Then the process $(X_t)_{t\geq 0}$ can be obtained in terms of subordinate Brownian motion with respect to the $(\alpha/2)$-stable subordinator:

$$X_t : \Omega_{\mathbf{P}_W} \times \Omega_{\mathbf{P}_S} \ni (\omega, \tau) \longmapsto B_{S_t(\tau)}(\omega) =: X_t(\omega, \tau).$$

This can also be seen by the composition of the characteristic exponent (2.3) with the Laplace exponent (2.2) which gives (2.1). Furthermore, $\mathbf{P}$ can then be identified as the image measure of this process on either of the spaces $D_{\rm r}(\mathbf{R}^+;\mathbf{R}^d)$ or $D_{\rm l}(\mathbf{R}^+;\mathbf{R}^d)$ such that

$$\mathbf{P}^x(X_t \in A) = (\mathbf{P}^x_W \times \mathbf{P}^0_S)(B_{S_t} \in A)$$

holds for all Borel sets $A \subset \mathbf{R}^d$.

The transition density $p(t,x)$ of the the process $(X_t)_{t\geq 0}$ is a smooth real-valued function on $\mathbf{R}^d$ determined by

$$\int_{\mathbf{R}^d} p(t,z)e^{iz\xi}dz = e^{-t|\xi|^\alpha}, \quad \xi \in \mathbf{R}^d,\ t>0.$$

We have $\mathbf{P}^x(X_t \in A) = \int_A p(t,y-x)dy$ for every Borel set $A \in \mathbf{R}^d$. For each fixed $t>0$ the density $p(t,x)$ is strictly positive, continuous and bounded on $\mathbf{R}^d$ satisfying the estimates

$$C^{-1}\left(\frac{t}{|x|^{d+\alpha}} \wedge t^{-d/\alpha}\right) \leq p(t,x) \leq C\left(\frac{t}{|x|^{d+\alpha}} \wedge t^{-d/\alpha}\right). \tag{2.4}$$

Note that for every $\alpha \in (0,2)$ the scaling property

$$p(t,x) = t^{-d/\alpha}p(1,t^{-1/\alpha}x), \quad x \in \mathbf{R}^d, \quad t>0$$

holds.

2.2. Two-sided stable processes

For our purposes below it will be important to consider stable processes $(X_t)_{t\geq 0}$ extended over the whole time-line $\mathbf{R}$ instead of defining them only on $\mathbf{R}^+$. Consider the measurable space $(\mathcal{Y}, \mathcal{B}(\mathcal{Y}))$, with $\mathcal{Y} = D_{\mathrm{r}}(\mathbf{R};\mathbf{R}^d)$, as well as $\tilde{D} = D_{\mathrm{r}} \times D_{\mathrm{l}}$ and $\mu^x = \mathbf{P}^x \times \mathbf{P}^x$. Let $\omega = (\omega_1,\omega_2) \in \tilde{D}$ and define

$$\tilde{X}_t(\omega) = \begin{cases} \omega_1(t), & t \geq 0, \\ \omega_2(-t), & t<0. \end{cases}$$

Since $\tilde{X}_t(\omega)$ is càdlàg in $t \in \mathbf{R}$ under μ^x, $X : (\tilde{D}, \mathcal{B}(\tilde{D})) \to (\mathcal{Y}, \mathcal{B}(\mathcal{Y}))$ can be defined by $X_t(\omega) = \tilde{X}_t(\omega)$. It is seen that $X \in \mathcal{B}(\tilde{D})/\mathcal{B}(\mathcal{Y})$ by showing that $X^{-1}(E) \in \mathcal{B}(\tilde{D})$, for any cylinder sets $E \in \mathcal{B}(\mathcal{Y})$. Thus X is a $\mathcal{Y}$-valued random variable on $\tilde{D}$. Denote the image measure of μ^x on $(\mathcal{Y}, \mathcal{B}(\mathcal{Y}))$ by $\tilde{\mathbf{P}}^x$. The coordinate process denoted by the same symbol

$$\tilde{X}_t : \omega \in \mathcal{Y} \mapsto \omega(t) \in \mathbf{R}^d \tag{2.5}$$

is an *α-stable process over* $\mathbf{R}$ on $(\mathcal{Y}, \mathcal{B}(\mathcal{Y}), \tilde{\mathbf{P}}^x)$ with $\tilde{X}_0 = x$ almost surely. The properties of stable processes on the whole real line can be summarized as follows.

Proposition 2.1. *The following hold:*

1. $\tilde{\mathbf{P}}^x(\tilde{X}_0 = x) = 1$;
2. *the increments* $(\tilde{X}_{t_i} - \tilde{X}_{t_{i-1}})_{1\leq i\leq n}$ *are independent symmetric α-stable random variables for any* $0 = t_0 < t_1 < \cdots < t_n$ *with* $\tilde{X}_t - \tilde{X}_s \stackrel{\mathrm{d}}{=} X_{t-s}$ *for* $t>s$;
3. *the increments* $(\tilde{X}_{-t_{i-1}} - \tilde{X}_{-t_i})_{1\leq i\leq n}$ *are independent symmetric α-stable random variables for any* $0 = -t_0 > -t_1 > \cdots > -t_n$ *with* $\tilde{X}_{-t} - \tilde{X}_{-s} \stackrel{\mathrm{d}}{=} X_{s-t}$ *for* $-t > -s$;
4. *the function* $\mathbf{R} \ni t \mapsto \tilde{X}_t(\omega) \in \mathbf{R}$ *is càdlàg for almost every* ω;
5. X_t *and* X_s *for* $t>0$ *and* $s<0$ *are independent.*

It can be checked directly through the finite dimensional distributions that the joint distribution of $\tilde{X}_{t_0}, \ldots, \tilde{X}_{t_n}$, $-\infty < t_0 < t_1 < \ldots < t_n < \infty$ with respect to $dx \otimes d\tilde{\mathbf{P}}^x$ is invariant with respect to time shift, i.e.,

$$\int dx \tilde{\mathbf{E}}^x\left[\prod_{i=0}^n f_i(\tilde{X}_{t_i})\right] = \int dx \tilde{\mathbf{E}}^x\left[\prod_{i=0}^n f_i(\tilde{X}_{t_i+s})\right]$$

for all $s \in \mathbf{R}$. Moreover, the left hand side of above formula can be expressed in terms of $\mathbf{P}^x$ on D_{r} as

$$\int dx \tilde{\mathbf{E}}^x \left[\prod_{i=0}^n f_i(\tilde{X}_{t_i}) \right] = \int dx \mathbf{E}^x \left[\prod_{i=0}^n f_i(X_{t_i - t_0}) \right].$$

Generally, we will consider the process $(X_t)_{t \in \mathbf{R}}$ starting at an arbitrary time $s \in \mathbf{R}$. For $s, t \in \mathbf{R}$ and $x, y \in \mathbf{R}^d$ we denote its transition density by

$$p(s, x, t, y) = \begin{cases} p(t-s, y-x) & \text{for} \quad s < t \\ 0 & \text{for} \quad s \geq t. \end{cases}$$

By $\mathbf{P}^{s,x}$ and $\mathbf{E}^{s,x}$ we respectively denote the distribution and expectation of the process $(X_t)_{t \geq s}$ starting at the point $x \in \mathbf{R}^d$ at time $s \in \mathbf{R}$. We have

$$\mathbf{P}^{s,x}(X_t \in A) = \int_A p(s, x, t, y) dy,$$

where by X_t we mean the canonical right continuous coordinate process evaluated at time $t > s$, and $A \in \mathbf{R}^d$ is a Borel set. When $s = 0$, we simply write $\mathbf{P}^x$ and $\mathbf{E}^x$ as before. The following time translation and scaling properties hold:

$$(X_t, \mathbf{P}^{s,x}) \stackrel{d}{=} (X_{t-s}, \mathbf{P}^x), \qquad (X_t, \mathbf{P}^{s,x}) \stackrel{d}{=} (r X_{r^{-\alpha} t}, \mathbf{P}^{s r^{-\alpha}, x r^{-1}}), \quad r > 0.$$

2.3. **Basic facts of potential theory for stable processes**

We now summarize the properties of the symmetric α-stable process and some facts from its potential theory which we will use below. For $x \in \mathbf{R}^d$, $|x|$ denotes the Euclidean norm of x. By $B(x, r)$, $x \in \mathbf{R}^d$, $r > 0$, we denote the standard Euclidean open ball of radius r centred in x. The set U^c is the complement in $\mathbf{R}^d$ of an arbitrary subset $U \subset \mathbf{R}^d$. For a set U and $r > 0$ we also define $rU = \{rx : x \in U\}$. By C_κ we mean a strictly positive and finite constant depending on α, d and parameter κ (for simplicity we omit α and d in the notation). We adopt the convention that constants may change their values from one use to the next. On occasion we write $C_\kappa^{(1)}, C_\kappa^{(2)}$ etc to distinguish between constants.

$(X_t, \mathbf{P}^x)$ is a standard rotation invariant α-stable Lévy process with Lévy measure given by the density

$$\nu(x) = \mathcal{A}_{d,-\alpha} |x|^{-d-\alpha},$$

where $\mathcal{A}_{d,\gamma} = 2^{-\gamma} \pi^{-d/2} \Gamma((d-\gamma)/2) |\Gamma(\gamma/2)|^{-1}$. For the remainder of the paper we will simply write $\mathcal{A}$ instead of $\mathcal{A}_{d,-\alpha}$.

Denote

$$P_t f(x) := \mathbf{E}^x[f(X_t)] = \int_{\mathbf{R}^d} f(y) p(t, x-y) dy.$$

Using the estimates (2.4), it is easy to show that the operators $P_t : L^1(\mathbf{R}^d) \to L^\infty(\mathbf{R}^d)$, $P_t : L^1(\mathbf{R}^d) \to L^1(\mathbf{R}^d)$, $P_t : L^\infty(\mathbf{R}^d) \to L^\infty(\mathbf{R}^d)$ and $P_t : L^2(\mathbf{R}^d) \to L^2(\mathbf{R}^d)$ are bounded.

It is known that when $\alpha < d$, then the process X_t is transient and the potential kernel for X_t is given by ([8, 43])

$$\Pi_\alpha(y-x) = \int_0^\infty p(t, y-x) dt = \mathcal{A}_{d,\alpha} |y-x|^{\alpha-d}, \quad x, y \in \mathbf{R}^d.$$

Whenever $\alpha \geq d$, the process is recurrent (pointwise recurrent when $\alpha > d = 1$). In this case we can consider the compensated potential kernel (see [9]), i.e., for $\alpha \geq d$ we put

$$\Pi_\alpha(y-x) = \int_0^\infty (p(t, y-x) - p(t, x_0))\, dt,$$

where $x_0 = 0$ for $\alpha > d = 1$ and $x_0 = 1$ for $\alpha = d = 1$. In this case

$$\Pi_\alpha(x) = \frac{1}{\pi} \log \frac{1}{|x|}$$

for $\alpha = d = 1$ and

$$\Pi_\alpha(x) = (2\Gamma(\alpha)\cos(\pi\alpha/2))^{-1}|x|^{\alpha-1}, \quad x \in \mathbf{R}^d$$

for $\alpha > d = 1$.

We denote by $p_D(t, x, y)$ the transition density of the process killed on exiting an open set $D \subset \mathbf{R}^d$. It satisfies the relation

$$p_D(t, x, y) = p(t, y-x) - \mathbf{E}^x[\tau_D \leq t; p(t - \tau_D, y - X_{\tau_D})], \quad x, y \in D,\, t > 0,$$

where $\tau_D = \inf\{t > 0 : X_t \notin D\}$ is the first exit time from D. For every fixed $t > 0$, the kernel $p_D(t, x, y)$ is strictly positive on $D \times D$. We put $p_D(t, x, y) = 0$ whenever $x \notin D$ or $y \notin D$. It is clear that $\mathbf{P}^x(\tau_D > t) = \int_D p_D(t, x, y)dy$, $x \in D$.

The Green function of an open bounded set D is defined by

$$G_D(x, y) = \int_0^\infty p_D(t, x, y)dt.$$

For a non-negative Borel function f on $\mathbf{R}^d$ we have

$$\int_D G_D(x, y)f(y)dy = \mathbf{E}^x\left[\int_0^{\tau_D} f(X_t)dt\right].$$

We will often use below the fact that

$$\mathbf{E}^x[\tau_{B(0,r)}] = c(r^2 - |x|^2)^{\alpha/2}, \tag{2.6}$$

$r > 0$, $x \in B(0, r)$, $c = \Gamma(d/2)(2^\alpha\Gamma(1 + \alpha/2)\Gamma((d + \alpha)/2))^{-1}$, see [31]. For a more thorough introduction to the potential theory of stable processes we refer to [10, 39, 19, 13].

2.4. **Stable bridges**

For $x, y \in \mathbf{R}^d$ and $s, t \in \mathbf{R}$, $s < t$, we respectively denote by $\mathbf{P}^{x,y}_{s,t}$ and $\mathbf{E}^{x,y}_{s,t}$ the distribution and expectation of the symmetric α-stable bridge $(X_r)_{0 \leq r < t}$ starting at the point $x \in \mathbf{R}^d$ at time $s \in \mathbf{R}$ given by $\lim_{r \nearrow t} X_r = y$ (see [30, Prop. 1], [4, Sect. VIII.3] and [51]). In fact, $(\mathbf{P}^{x,y}_{s,t})_{y \in \mathbf{R}^d}$ is a regular version of the family of conditional probability distributions $\mathbf{P}^{s,x}(\,\cdot\,|X_t = y)$, $y \in \mathbf{R}^d$, that is, if $Y \geq 0$ is $\mathcal{F}_{[s,t)}$-measurable and $g \geq 0$ is a Borel function on $\mathbf{R}^d$, then (see [30, (2.8)])

$$\mathbf{E}^{s,x}(Y g(X_t)) = \int_{\mathbf{R}^d} \mathbf{E}^{x,y}_{s,t}(Y) g(y) p(t - s, y - x)dy.$$

Under $\mathbf{P}^{x,y}_{s,t}$, the coordinate process $(X_r)_{0 \leq r < t}$ is a non-homogenous strongly Markov process with transition densities

$$p_{t,y}(u, a, v, b) = \frac{p(t - u, b - a)p(t - v, y - b)}{p(t - u, y - a)}, \quad s \leq u < v < t\,.$$

Clearly, $\mathbf{P}^{x,y}_{s,t}(X_s = x, \lim_{r \nearrow t} X_r = y) = 1$.

For $x, y \in \mathbf{R}^d$ and $s, t \in \mathbf{R}$, $s < t$, we denote by $\nu^{x,y}_{[s,t)}$ the measure on $(\Omega_{[s,t)}, \mathcal{F}_{[s,t)})$ corresponding to the symmetric α-stable bridge $(X_r : s \leq r < t)$ (see (2.5), the remarks below and [30, (2.6)]) given by

$$\nu^{x,y}_{[s,t)}(\,\cdot\,) = p(t-s, y-x)\mathbf{P}^{x,y}_{s,t}(\,\cdot\,). \tag{2.7}$$

Thus for $s = t_0 < t_1 < t_2 < ... < t_n < t_{n+1} = t$ and Borel sets $A_1, A_2, ..., A_n \subset \mathbf{R}^d$ we have

$$\begin{aligned} &\nu^{x,y}_{[s,t)}(\omega(t_1) \in A_1, \omega(t_2) \in A_2, ..., \omega(t_n) \in A_n) \\ &= \int_{A_1} ... \int_{A_n} \prod_{i=1}^{n+1} p(t_i - t_{i-1}, z_i - z_{i-1}) dz_1 ... dz_n, \end{aligned} \tag{2.8}$$

where $z_0 = x$ and $z_{n+1} = y$.

Since $\nu^{x,y}_{[s,t)}$ is the measure defined on the set of right continuous paths with left limits, we may also identify $\nu^{x,y}_{[s,t)}$ with a measure on $(\Omega, \mathcal{F}_{[s,t)})$. We will write $\nu^{x,y}_T$ instead of $\nu^{x,y}_{[-T,T)}$, $T > 0$.

3. **Fractional Schrödinger semigroups**

3.1. **Fractional Schrödinger operator and its Feynman-Kac semigroup**

The $L^2(\mathbf{R}^d)$-generator of the semigroup $\{P_t : t \geq 0\}$ is the fractional Laplace operator $-(-\Delta)^{\alpha/2}$, $\alpha \in (0,2)$. Recall that the operator with domain $H^\alpha(\mathbf{R}^d) = \{f \in L^2(\mathbf{R}^d) : |k|^\alpha \hat{f} \in L^2(\mathbf{R}^d)\}$, $0 < \alpha < 2$, defined by

$$\widehat{(-\Delta)^{\alpha/2} f}(k) = |k|^\alpha \hat{f}(k),$$

is the *fractional Laplacian* of order $\alpha/2$. It is essentially self-adjoint on $C_0^\infty(\mathbf{R}^d)$, and its spectrum is $\mathrm{Spec}((-\Delta)^{\alpha/2}) = \mathrm{Spec}_{\mathrm{ess}}((-\Delta)^{\alpha/2}) = [0, \infty)$.

Let $V : \mathbf{R}^d \to \mathbf{R}$ be a Borel measurable function. We call V *potential* and view it as a multiplication operator to define fractional Schrödinger operators by choosing it from a suitable function space.

We define the space of potentials we will consider.

Definition 3.1. (Fractional Kato-class) We say that the Borel function $V : \mathbf{R}^d \to \mathbf{R}$ belongs to the *fractional Kato-class* $\mathcal{K}^\alpha$ if V satisfies either of the two equivalent conditions (see [57] and (2.5) in [12])

$$\lim_{\epsilon \to 0} \sup_{x \in \mathbf{R}^d} \int_{|y-x|<\epsilon} |V(y)\Pi_\alpha(y-x)| dy = 0,$$

$$\lim_{t \to 0} \sup_{x \in \mathbf{R}^d} \int_0^t P_u|V|(x) du = 0.$$

We write $V \in \mathcal{K}^\alpha_{\mathrm{loc}}$ if $V\mathbf{1}_B \in \mathcal{K}^\alpha$ for every ball $B \subset \mathbf{R}^d$. Moreover, we say that V is a *fractional Kato-decomposable potential* whenever

$$V = V_+ - V_- \quad \text{with} \quad V_- \in \mathcal{K}^\alpha, \;\; V_+ \in \mathcal{K}^\alpha_{\mathrm{loc}},$$

where V_+ and V_- denote the positive and negative parts of V, respectively.

For simplicity, in what follows we omit the qualifier "fractional" in the use of this terminology.

Example 3.1. Some examples and counterexamples of Kato-potentials are as follows.

(1) *Locally bounded potentials:* Let $V \in L^\infty_{\text{loc}}$. Then for all $\alpha \in (0,2)$ we have $V \in \mathcal{K}^\alpha_{\text{loc}}$ and V is Kato-decomposable.
(2) *Potentials with local singularities:* Let $V(x) = \pm|x|^{-\beta}$, $\beta > 0$. Then $V \in \mathcal{K}^\alpha$ if and only if $\beta < \alpha$. Let $k \in \mathbf{N}$, $x_i \in \mathbf{R}^d$, $\beta_i > 0$ and $\varepsilon_i \in \{-1,1\}$ for $1 \le i \le k$. Then the potential
$$V(x) = \sum_{i=1}^k \varepsilon_i |x - x_i|^{-\beta_i}$$
belongs to $\mathcal{K}^\alpha$ whenever each $\beta_i < \alpha$.
(3) *Coulomb potential:* Let $d = 3$. In the light of (2) above the Coulomb potential $V(x) = -\frac{C}{|x|}$ belongs to Kato-class $\mathcal{K}^\alpha$ for $\alpha \in (1,2)$ only.

Definition 3.2. (Fractional Schrödinger operator for bounded potential) If $V \in L^\infty(\mathbf{R}^d)$ we call

$$H_\alpha := (-\Delta)^{\alpha/2} + V, \quad 0 < \alpha < 2 \tag{3.1}$$

fractional Schrödinger operator with potential V. We call the one-parameter operator semigroup $\{e^{-tH_\alpha} : t \ge 0\}$ *fractional Schrödinger semigroup.*

For our purposes below it is important to know that the spectrum of the fractional Schrödinger operator is a subset of the real line. As long as V is relatively (resp. form) bounded with respect to the fractional Laplacian, the Kato-Rellich theorem (resp. KLMN theorem) applies (see [33, 47]) and the operator H_α is a self-adjoint perturbation of the fractional Laplacian, defined as an operator (resp. form) sum of two self-adjoint operators, and therefore Spec $H_\alpha \subset \mathbf{R}$. In a next step (Theorem 3.2 below) we will define fractional Schrödinger operators for Kato-decomposable potentials V as self-adjoint operators in which V need not be a perturbation.

The following theorem states that a Feynman-Kac-type formula for fractional Schrödinger operators holds.

Theorem 3.1. (Functional integral representation) *Let $V \in L^\infty(\mathbf{R}^d)$, and $f, g \in L^2(\mathbf{R}^d)$. We have*

$$(f, e^{-t\left((-\Delta)^{\alpha/2}+V\right)} g) = \int_{\mathbf{R}^d} dx \mathbf{E}^x \left[\overline{f(X_0)} g(X_t) e^{-\int_0^t V(X_s) ds} \right]. \tag{3.2}$$

On the left hand side the bracket denotes L^2-scalar product.

Proof. We divide the proof into four steps.

(Step 1) Suppose that $V \equiv 0$. Our first claim is

$$(f, e^{-t\left((-\Delta)^{\alpha/2}\right)} g) = \int_{\mathbf{R}^d} dx \mathbf{E}^x \left[\overline{f(X_0)} g(X_t) \right]. \tag{3.3}$$

To prove (3.3) it is convenient to regard the process $(X_t)_{t\ge0}$ as the composition of Brownian motion $(B_t)_{t\ge0}$ and the $(\alpha/2)$-stable subordinator $(S_t)_{t\ge0}$ as explained above. Let E denote the spectral projection of the self-adjoint operator $-\Delta \ge 0$. Then by using (2.2) and the usual Feynman-Kac formula for $e^{t\Delta}$ we have

$$\begin{aligned}\left(f, e^{-t(-\Delta)^{\alpha/2}}g\right) &= \int_0^\infty e^{-t\lambda^{\alpha/2}} d(f, E_\lambda g) = \int_0^\infty \mathbf{E}^0_{\mathbf{P}_S}\left[e^{-\lambda S_t}\right] d(f, E_\lambda g)\\ &= \mathbf{E}^0_{\mathbf{P}_S}\left[\int_0^\infty e^{-S_t\lambda} d(f, E_\lambda g)\right] = \mathbf{E}^0_{\mathbf{P}_S}\left[(f, e^{-S_t(-\Delta)}g)\right]\\ &= \mathbf{E}^0_{\mathbf{P}_S}\left[\int_{\mathbf{R}^d} dx \mathbf{E}^x_{\mathbf{P}_W}[\overline{f(B_0)}g(B_{S_t})]\right] = \int_{\mathbf{R}^d} dx \mathbf{E}^x[\overline{f(X_0)}g(X_t)],\end{aligned}$$

thus (3.3) follows.

(Step 2) Let $0 = t_0 < t_1 < ... < t_n$, $f_0, f_n \in L^2(\mathbf{R}^d)$ and assume that $f_j \in L^\infty(\mathbf{R}^d)$, for $j = 1, 2, ..., n-1$. We claim that

$$\left(f_0, \prod_{j=1}^n e^{-(t_j - t_{j-1})(-\Delta)^{\alpha/2}} f_j\right) = \int_{\mathbf{R}^d} dx \mathbf{E}^x\left[\overline{f(X_0)} \prod_{j=1}^n f_j(X_{t_j})\right]. \tag{3.4}$$

For simplifying the notation put $s_j = t_j - t_{j-1}$, for any $j = 1, ..., n$ and

$$g_i = f_i \left(\prod_{j=i+1}^n e^{-s_j(-\Delta)^{\alpha/2}} f_j\right), \quad j = 1, ...n-1,\ g_n = f_n.$$

Notice that $g_j = f_j e^{-s_{j+1}(-\Delta)^{\alpha/2}} g_{j+1}$. By (3.3) the left hand side of (3.4) can be represented as

$$\int_{\mathbf{R}^d} dx \mathbf{E}^x\left[\overline{f(X_0)} g_1(X_{s_1})\right] = \int_{\mathbf{R}^d} dx \overline{f(x)} \mathbf{E}^x\left[g_1(X_{s_1})\right].$$

Using (3.3) again, we obtain

$$\begin{aligned}\mathbf{E}^x\left[g_j(X_{s_j})\right] &= \int_{\mathbf{R}^d} p(s_j, y-x) g_j(y) dy = \int_{\mathbf{R}^d} p(s_j, y-x) f_j(y) e^{-s_{j+1}(-\Delta)^{\alpha/2}} g_{j+1}(y) dy\\ &= \int_{\mathbf{R}^d} \mathbf{E}^y\left[p(s_j, X_0 - x) f_j(X_0) g_{j+1}(X_{s_{j+1}})\right] dy\\ &= \int_{\mathbf{R}^d} p(s_j, y-x) f_j(y) \mathbf{E}^y\left[g_{j+1}(X_{s_{j+1}})\right] dy = \mathbf{E}^x\left[f_j(X_{s_j}) \mathbf{E}^{X_{s_j}}\left[g_{j+1}(X_{s_{j+1}})\right]\right],\end{aligned}$$

for $j = 1, ..., n-1$. The above equalities yield

$$\begin{aligned}\left(f_0, \prod_{j=1}^n e^{-s_j(-\Delta)^{\alpha/2}} f_j\right) &= \int_{\mathbf{R}^d} dx \mathbf{E}^x\left[\overline{f(X_0)} f_1(X_{s_1}) \times\right.\\ &\left.\times \mathbf{E}^{X_{s_1}}\left[f_2(X_{s_2}) \mathbf{E}^{X_{s_2}}\left[f_3(X_{s_3}) \mathbf{E}^{X_{s_3}}\left[\dots \mathbf{E}^{X_{s_{n-1}}}\left[f_n(X_{s_n})\right]\dots\right]\right]\right]\right],\end{aligned}$$

and (3.4) follows by the Markov property of $(X_t)_{t\geq 0}$.

(Step 3) Let now $0 \neq V \in C_b(\mathbf{R}^d)$. We show (3.2) for such V. Since $(-\Delta)^{\alpha/2}$ is self-adjoint on $D((-\Delta)^{\alpha/2})$, the Trotter product formula holds:

$$\left(f, e^{-t((-\Delta)^{\alpha/2}+V)}g\right) = \lim_{n\to\infty}\left(f, \left(e^{-(t/n)(-\Delta)^{\alpha/2}} e^{-(t/n)V}\right)^n g\right).$$

Combined with (Step 2) it yields

$$\left(f, e^{-t((-\Delta)^{\alpha/2}+V)}g\right) = \lim_{n\to\infty} \int_{\mathbf{R}^d} dx \mathbf{E}^x\left[\overline{f(X_0)} g(X_t) e^{-\sum_{j=1}^n (t/n) V(X_{tj/n})}\right].$$

Since each càdlàg path $s \mapsto \omega(s) = X_s(\omega)$ is continuous in $s \in [0,t]$ except for at most countably many points, we have $\sum_{j=1}^n (t/n)V(X_{tj/n}) \to \int_0^t V(X_s)ds$ as $n \to \infty$ in the sense of Riemann integral. Thus (3.2) follows for $V \in C_b(\mathbf{R}^d)$.

(Step 4) We make use of the argument in [55, Th. 6.2] to complete the proof. Suppose that $V \in L^\infty(\mathbf{R}^d)$ and let $V_n = \phi(x/n)(V * h_n)$, where $h_n = n^d\phi(nx)$ with $\phi \in C_0^\infty(\mathbf{R}^d)$ such that $0 \le \phi \le 1$, $\int \phi(x)dx = 1$ and $\phi(0) = 1$. Then $V_n \to V$ almost everywhere and V_n are bounded and continuous. Let $\mathcal{N}$ denote the set of all x such that $V_n(x)$ does not converge to $V(x)$. Then the measure of $\{t \in [0,\infty) : X_t(\omega) \in \mathcal{N}\}$ is zero $\mathbf{P}^x$-almost surely and $\int_0^t V_n(X_s)ds \to \int_0^t V(X_s)ds$ as $n \to \infty$ $\mathbf{P}^x$-a.s. Thus

$$\int_{\mathbf{R}^d} dx \mathbf{E}^x \left[\overline{f(X_0)} g(X_t) e^{-\int_0^t V_n(X_s)ds}\right] \to \int_{\mathbf{R}^d} dx \mathbf{E}^x \left[\overline{f(X_0)} g(X_t) e^{-\int_0^t V(X_s)ds}\right]$$

as $n \to \infty$. On the other hand, $e^{-t((-\Delta)^{\alpha/2}+V_n)} \to e^{-t((-\Delta)^{\alpha/2}+V)}$ in strong sense as $n \to \infty$, since $(-\Delta)^{\alpha/2} + V_n \to (-\Delta)^{\alpha/2} + V$ on the joint domain $D((-\Delta)^{\alpha/2})$. □

Theorem 3.1 above can be generalized to Schrödinger-type operators obtained as the (operator or form) sum of a Bernstein function of the Laplacian and a potential V. For further details see [33, Th. 3.8].

Using Theorem 3.1, we define the *Feynman-Kac functional* for the symmetric α-stable process by

$$e_V(t) := e_V(t)(\omega) = e^{-\int_0^t V(X_s(\omega))ds}, \quad t > 0.$$

If $V \in \mathcal{K}^\alpha$, then there are constants $C_V^{(0)}$, $C_V^{(1)}$ such that (see [12, (2.10)])

$$\sup_{x \in \mathbf{R}^d} \mathbf{E}^x[e_{-|V|}(t)] \le e^{C_V^{(0)} + C_V^{(1)} t}. \tag{3.5}$$

When V is Kato-decomposable, then clearly $e_V(t) \le e_{-V_-}(t)$, and therefore

$$\sup_{x \in \mathbf{R}^d} \mathbf{E}^x[e_V(t)] \le e^{C_{V_-}^{(0)} + C_{V_-}^{(1)} t}. \tag{3.6}$$

Clearly, V_+ has a killing effect and V_- has a mass generating effect in the Feynman-Kac functional.

Theorem 3.2. (Feynman-Kac semigroup) *Let V be a Kato-decomposable potential and*

$$(T_t f)(x) := \mathbf{E}^x \left[e_V(t) f(X_t)\right], \quad t \ge 0.$$

Then $\{T_t : t \ge 0\}$ is a strongly continuous symmetric semigroup. In particular, there exists a self-adjoint operator H bounded from below such that $e^{-tH} = T_t$.

$\{T_t : t \ge 0\}$ is called *Feynman-Kac semigroup* associated with the symmetric α-stable process.

Proof. Let $V = V_+ - V_-$. Using (3.5) we have

$$\begin{aligned} \|T_t f\|^2 &\le \int_{\mathbf{R}^d} dx \mathbf{E}^x \left[e^{-2\int_0^t V_+(X_s)ds} |f(X_t)|^2\right] \mathbf{E}^x \left[e^{2\int_0^t V_-(X_s)ds}\right] \\ &\le C_t \int_{\mathbf{R}^d} dx \mathbf{E}^x \left[|f(X_t)|^2\right] \\ &= C_t \|e^{(t/2)\Delta} f\|^2 \le C_t \|f\|^2, \end{aligned}$$

where $C_t = \sup_{x \in \mathbf{R}^d} \mathbf{E}^x[e^{2\int_0^t V_-(X_s)ds}]$. Thus T_t is a bounded operator from $L^2(\mathbf{R}^d)$ to $L^2(\mathbf{R}^d)$. Similarly as in Step 2 of the proof of Theorem 3.1 it is seen that the semigroup property $T_t T_s = T_{t+s}$ holds for $t, s \ge 0$.

To check strong continuity of T_t in t, it suffices to show weak continuity. Let $f, g \in C_0^\infty(\mathbf{R}^d)$. Then we have

$$(f, T_t g) = \int_{\mathbf{R}^d} dx \mathbf{E}^{x,0}_{\mathbf{P}_W \times \mathbf{P}_S} \left[\overline{f(B_0)} g(B_{S_t}) e^{-\int_0^t V(B_{S_r}) dr} \right].$$

Since $S_t(\tau) \to 0$ as $t \to 0$ for each $\tau \in \Omega_{\mathbf{P}_S}$, dominated convergence gives $(f, T_t g) \to (f, g)$.

To conclude, we check the symmetry property $T_t^* = T_t$. Let $\widetilde{B}_s = B_{S_t(\tau)-s}(\omega) - B_{S_t(\tau)}(\omega)$. Then for each $\tau \in \Omega_{\mathbf{P}_S}$, $\widetilde{B}_s \stackrel{\rm d}{=} B_s$ with respect to $\mathbf{P}^x_W$. ($Z \stackrel{\rm d}{=} Y$ denotes that Z and Y are identically distributed.) Thus we have

$$\begin{aligned} (f, T_t g) &= \int_{\mathbf{R}^d} dx \overline{f(x)} \mathbf{E}^{x,0}_{\mathbf{P}_W \times \mathbf{P}_S} \left[e^{-\int_0^t V(\widetilde{B}_{S_r}) dr} g(\widetilde{B}_{S_t}) \right] \\ &= \mathbf{E}^{0,0}_{\mathbf{P}_W \times \mathbf{P}_S} \left[\int_{\mathbf{R}^d} dx \overline{f(x)} e^{-\int_0^t V(x + \widetilde{B}_{S_r}) dr} g(x + \widetilde{B}_{S_t}) \right] \\ &= \mathbf{E}^{0,0}_{\mathbf{P}_W \times \mathbf{P}_S} \left[\int_{\mathbf{R}^d} dx \overline{f(x - \widetilde{B}_{S_t})} e^{-\int_0^t V(x + \widetilde{B}_{S_r} - \widetilde{B}_{S_t}) dr} g(x) \right]. \end{aligned}$$

In the second equality we changed the variable x to $x - \widetilde{B}_{S_t}$. Since $\widetilde{B}_{S_t} \stackrel{\rm d}{=} -B_{S_t}$ and $\widetilde{B}_{S_r} - \widetilde{B}_{S_t} \stackrel{\rm d}{=} B_{S_t - S_r}$, we have

$$(f, T_t g) = \int_{\mathbf{R}^d} dx \mathbf{E}^{x,0}_{\mathbf{P}_W \times \mathbf{P}_S} \left[\overline{f(B_{S_t})} e^{-\int_0^t V(B_{S_t - S_r}) dr} g(x) \right].$$

Moreover, as $S_t - S_r \stackrel{\rm d}{=} S_{t-r}$ for $0 \leq r \leq t$, we obtain

$$\begin{aligned} (f, T_t g) &= \int_{\mathbf{R}^d} dx \mathbf{E}^{x,0}_{\mathbf{P}_W \times \mathbf{P}_S} \left[\overline{f(B_{S_t})} e^{-\int_0^t V(B_{S_{t-r}}) dr} g(x) \right] \\ &= \int_{\mathbf{R}^d} dx \overline{\mathbf{E}^{x,0}_{\mathbf{P}_W \times \mathbf{P}_S} \left[f(B_{S_t}) e^{-\int_0^t V(B_{S_r}) dr} \right]} g(x) \\ &= (T_t f, g). \end{aligned}$$

The existence of a self-adjoint operator H bounded from below such that $T_t = e^{-tH}$ follows now by the Hille-Yoshida theorem. This completes the proof. □

The above theorem allows to define H_α as a self-adjoint operator for cases when V is Kato-decomposable.

Definition 3.3. (Fractional Schrödinger operator for Kato-class) Let V be a Kato decomposable potential. We call H given by Theorem 3.2 a *fractional Schrödinger operator for Kato-decomposable potential* V. We refer to the one-parameter operator semigroups $\{e^{-tH_\alpha} : t \geq 0\}$ and $\{T_t : t \geq 0\}$ as the *fractional Schrödinger semigroup* and *Feynman-Kac semigroup* with Kato-decomposable potential V, respectively.

3.2. **Basic properties of fractional Feynman-Kac semigroups**

Using Kato-decomposable potentials has the benefit of allowing very good regularity properties of the corresponding Feynman-Kac semigroup. In this section we discuss the properties of its kernel.

Proposition 3.1. (L^p-boundedness) *Let V be a Kato-decomposable potential. Then each T_t is a bounded operator from $L^p(\mathbf{R}^d)$ to $L^q(\mathbf{R}^d)$, for all $1 \leq p \leq q \leq \infty$.*

Proof. [33, Th. 4.13]. □

Lemma 3.1. *Let V be a Kato-decomposable potential. For all $t > 0$, $x \in \mathbf{R}^d$ and every bounded set $A \subset \mathbf{R}^d$ with positive Lebesgue measure, we have $T_t \mathbf{1}_A(x) > 0$.*

Proof. Fix $x \in \mathbf{R}^d$, $t > 0$ and a set $A \subset \mathbf{R}^d$ with positive Lebesgue measure. Take a sufficiently large ball B such that $x \in B$ and $A \subset B$. Let $\mathbf{Q}^x = \mathbf{Q}^x_{A,B,t}$ be a probability measure given by the conditional probability

$$\mathbf{Q}^x_{A,B,t}(\,\cdot\,) = \mathbf{P}^x(\,\cdot\,|X_t \in A, t < \tau_B) = \frac{\mathbf{P}^x(\,\cdot\,, X_t \in A, t < \tau_B)}{\mathbf{P}^x(X_t \in A, t < \tau_B)}$$

with corresponding expectation $\mathbf{E}^x_{\mathbf{Q}}$. Clearly

$$\mathbf{P}^x(X_t \in A, t < \tau_B) = \int_A p_B(t,x,y)dy > 0.$$

Denote

$$\epsilon_{t,B,A} = \int_A p_B(t,x,y)dy.$$

Then we have

$$\begin{aligned}
T_t\mathbf{1}_A(x) &= \mathbf{E}^x[X_t \in A; e_V(t)] \geq \mathbf{E}^x[X_t \in A; e_{V_+}(t)] \\
&\geq \mathbf{E}^x[X_t \in A, t < \tau_B; e_{V_+}(t)] \\
&\geq \mathbf{E}^x[X_t \in A, t < \tau_B; e_{V_+}(\tau_B)] \\
&= \mathbf{P}^x(X_t \in A, t < \tau_B)\mathbf{E}^x_{\mathbf{Q}}[e_{V_+}(\tau_B)] \\
&\geq \epsilon_{t,B,A} \exp\left(-\mathbf{E}^x_{\mathbf{Q}}\left[\int_0^{\tau_B} V_+(X_s)ds\right]\right) \\
&\geq \epsilon_{t,B,A} \exp\left(-\frac{1}{\epsilon_{t,B,A}}\mathbf{E}^x\left[\int_0^{\tau_B} V_+(X_s)ds\right]\right) \\
&= \epsilon_{t,B,A} \exp\left(-\frac{\int_B G_B(x,y)V_+(y)dy}{\epsilon_{t,B,A}}\right)
\end{aligned}$$

by using the Jensen inequality. Since $V_+ \in \mathcal{K}^\alpha_{\mathrm{loc}}$, we have $V_+\mathbf{1}_B \in \mathcal{K}^\alpha$ and $\int_B G_B(\cdot,y)V_+(y)dy = \int_B G_B(\cdot,y)V_+\mathbf{1}_B(y)dy \in L^\infty(B)$. Thus $T_t\mathbf{1}_A(x) > 0$. $\square$

Next we state and prove the existence and basic properties of the kernel for the semigroup $\{T_t : t \geq 0\}$.

Theorem 3.3. *Let V be a Kato-decomposable potential. The following properties hold:*

1. *for every fixed $t > 0$ the operator T_t has a bounded integral kernel $u(t,x,y)$, i.e. $T_tf(x) = \int_{\mathbf{R}^d} u(t,x,y)f(y)dy$, $t > 0$, $x \in \mathbf{R}^d$, $f \in L^p(\mathbf{R}^d)$, $1 \leq p \leq \infty$;*
2. *$u(t,x,y) = u(t,y,x)$, for every $t > 0$, $x, y \in \mathbf{R}^d$;*
3. *for every $t > 0$, $u(t,x,y)$ is continuous on $\mathbf{R}^d \times \mathbf{R}^d$;*
4. *$u(t,x,y)$ is strictly positive on $(0,\infty) \times \mathbf{R}^d \times \mathbf{R}^d$;*
5. *for all $x, y \in \mathbf{R}^d$ and $s, t \in \mathbf{R}$, $s < t$, the functional representation (Feynman-Kac-type formula)*

$$u(t-s,x,y) = \int e^{-\int_s^t V(X_r(\omega))dr} d\nu^{x,y}_{[s,t)}(\omega), \tag{3.7}$$

holds, where the α-stable bridge measure $\nu^{x,y}_{[s,t)}$ is given by (2.8).

Proof. First consider (1). By Proposition 3.1 the operators $T_t : L^p(\mathbf{R}^d) \to L^\infty(\mathbf{R}^d)$, $1 \leq p \leq \infty$, are bounded for each $t > 0$. This together with the Dunford-Pettis theorem [54, Th. A.1.1, Cor. A.1.2] (compare also [21]) imply that a bounded measurable integral kernel $u(t, x, y)$ on $\mathbf{R}^d \times \mathbf{R}^d$ exists. Since for every fixed $t > 0$ the operator T_t is symmetric, (2) holds for almost all $(x, y) \in \mathbf{R}^d \times \mathbf{R}^d$ with respect to Lebesgue measure.

For the proof of (3) consider $\mathbf{R}^d \times \mathbf{R}^d$ endowed with the product topology, and the joint process $(\check{X}_t)_{t\geq 0} = ((X_t^{(1)})_{t\geq 0}, (X_t^{(2)})_{t\geq 0})$, where the two components are stochastically independent replicas of $(X_t)_{t\geq 0}$. For any given Kato-decomposable potential V define the semigroup $(\check{T}_t)$ by putting

$$\check{T}_t f(\check{x}) = \mathbf{E}^{\check{x}} \left[e_{\check{V}}(t) f(\check{X}_t) \right], \quad f \in L^\infty(\mathbf{R}^d \times \mathbf{R}^d),\ t \geq 0,$$

where $\check{x} = (x_1, x_2)$ and $\check{V}(\check{x}) = V(x_1) + V(x_2)$. An approximation by indicator functions over rectangles of the form $A_1 \times A_2$, $A_1, A_2 \in \mathbf{R}^d$, and standard arguments give that the semigroup $\{\check{T}_t : t \geq 0\}$ can be written as

$$\check{T}_t f(\check{x}) = \int_{\mathbf{R}^d\times\mathbf{R}^d} \check{u}(t, \check{x}, \check{y}) f(\check{y}) d\check{y} = \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} u(t, x_1, y_1) u(t, x_2, y_2) f(y_1, y_2) dy_1 dy_2$$

for $f \in L^\infty(\mathbf{R}^d \times \mathbf{R}^d)$, $t > 0$.

Now we show that $\check{T}_t : L^\infty(\mathbf{R}^d \times \mathbf{R}^d) \to C_b(\mathbf{R}^d \times \mathbf{R}^d)$. Put $V_n = \mathbf{1}_{B(0,n)} V_+ - V_-$, $n = 1, 2, ...$ By the assumption that $V_+ \in \mathcal{K}^\alpha_{\text{loc}}$ we have $V_n \in \mathcal{K}^\alpha$, $n = 1, 2, ...$ For any n we put $\check{T}_{n,t} f(\check{x}) = \mathbf{E}^{\check{x}} \left[e_{\check{V}_n}(t) f(\check{X}_t) \right]$, $t > 0$, $\check{x} \in \mathbf{R}^d \times \mathbf{R}^d$. Define the semigroup

$$\check{P}_t f(\check{x}) = \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} p(t, y_1 - x_1) p(t, y_2 - x_2) f(y_1, y_2) dy_1 dy_2, \quad f \in L^\infty(\mathbf{R}^d \times \mathbf{R}^d), \quad t \geq 0.$$

By the estimates (2.4) of the function $p(t, x)$ we get $\check{P}_t : L^\infty(\mathbf{R}^d \times \mathbf{R}^d) \to C_b(\mathbf{R}^d \times \mathbf{R}^d)$. Combined with (3.6) and the same argument as in [21, Prop. 3.11-3.12], we obtain that $\check{T}_{n,t} : L^\infty(\mathbf{R}^d \times \mathbf{R}^d) \to C_b(\mathbf{R}^d \times \mathbf{R}^d)$, for any $n = 1, 2...$ Furthermore,

$$\begin{aligned}
|\check{T}_t f(\check{x}) - \check{T}_{n,t} f(\check{x})| &= \left| \mathbf{E}^{\check{x}} [(e_{\check{V}}(t) - e_{\check{V}_n}(t)) f(\check{X}_t)] \right| \\
&\leq \|f\|_\infty \mathbf{E}^{\check{x}} \left[e_{\check{V}_n}(t) \left| e^{-\int_0^t \left(\mathbf{1}_{B^c(0,n)} V_+(X_s^{(1)}) + \mathbf{1}_{B^c(0,n)} V_+(X_s^{(2)}) \right) ds} - 1 \right| \right] \\
&\leq \|f\|_\infty \mathbf{E}^{\check{x}} \left[e_{-\check{V}_-}(t) \left(1 - e^{-\int_0^t \left(\mathbf{1}_{B^c(0,n)} V_+(X_s^{(1)}) + \mathbf{1}_{B^c(0,n)} V_+(X_s^{(2)}) \right) ds} \right) \right] \\
&= \|f\|_\infty \left(\mathbf{E}^{\check{x}} \left[\mathbf{1}_{A_n^c}\, e_{-\check{V}_-}(t) \left(1 - e^{-\int_0^t \left(\mathbf{1}_{B^c(0,n)} V_+(X_s^{(1)}) + \mathbf{1}_{B^c(0,n)} V_+(X_s^{(2)}) \right) ds} \right) \right] \right. \\
&\qquad \left. + \mathbf{E}^{\check{x}} \left[\mathbf{1}_{A_n}\, e_{-\check{V}_-}(t) \left(1 - e^{-\int_0^t \left(\mathbf{1}_{B^c(0,n)} V_+(X_s^{(1)}) + \mathbf{1}_{B^c(0,n)} V_+(X_s^{(2)}) \right) ds} \right) \right] \right) \\
&= \|f\|_\infty (g_{A_n^c}(\check{x}) + g_{A_n}(\check{x})),
\end{aligned}$$

where $A_n = \left\{ \omega : \tau_{B(0,n)}^{X_t^{(1)}} \geq t, \tau_{B(0,n)}^{X_t^{(2)}} \geq t \right\}$. Clearly, $g_{A_n}(\check{x}) = 0$ for $\check{x} \in B(0, n) \times B(0, n)$. Moreover, by Schwarz inequality and stochastic independence of the processes $(X_t^{(1)})_{t\geq 0}$ and $(X_t^{(2)})_{t\geq 0}$ we obtain

$$\begin{aligned} f_{B_n^c}(\check{x}) &\le \sum_{i=1}^2 \left(\mathbf{E}^{\check{x}} \left[\tau_{B(0,n)}^{X_t^{(i)}} < t : e_{-\check{V}_-}(t) \right] \right) \\ &\le \sum_{i=1}^2 \left(\left(\mathbf{P}^{\check{x}} (\tau_{B(0,n)}^{X_t^{(i)}} < t) \right)^{1/2} \left(\mathbf{E}^{\check{x}} [e_{-2\check{V}_-}(t)] \right)^{1/2} \right) \\ &= \left(\mathbf{E}^{x_1} [e_{-2V_-}(t)] \right)^{1/2} \left(\mathbf{E}^{x_2} [e_{-2V_-}(t)] \right)^{1/2} \sum_{i=1}^2 \left(\mathbf{P}^{x_i} (\tau_{B(0,n)}^{X_t^{(i)}} < t) \right)^{1/2} \\ &\le C_{V,t} \sum_{i=1}^2 \left(\mathbf{P}^{x_1} (\tau_{B(0,n)}^{X_t^{(i)}} < t) \right)^{1/2}. \end{aligned}$$

Since for each fixed $t > 0$ and $i = 1, 2$ we have $\mathbf{P}^{x_i}(\tau_{B(0,n)}^{X_t^{(i)}} < t) \to 0$ as $n \to \infty$, uniformly in x_i on each compact set, this gives that $|\check{T}_t f(\check{x}) - \check{T}_{n,t} f(\check{x})| \to 0$ as $n \to \infty$, uniformly in $\check{x}$ on every compact set. This and the fact that $\check{T}_{n,t} : L^\infty(\mathbf{R}^d \times \mathbf{R}^d) \to C_b(\mathbf{R}^d \times \mathbf{R}^d)$, $n = 1, 2, 3, \ldots$, imply that $\check{T}_t : L^\infty(\mathbf{R}^d \times \mathbf{R}^d) \to C_b(\mathbf{R}^d \times \mathbf{R}^d)$.

Thus for each fixed $t > 0$ we have $\check{T}_t u(t, x_1, x_2) \in C_b(\mathbf{R}^d \times \mathbf{R}^d)$. This shows that for every $t > 0$ the kernel $u(t, x, y)$ has a jointly continuous version on $\mathbf{R}^d \times \mathbf{R}^d$. Indeed, by the semigroup property and (2) for almost every x, y, we have for every function $g \in C_c(\mathbf{R}^d \times \mathbf{R}^d)$

$$\begin{aligned} &\int_{\mathbf{R}^d} \int_{\mathbf{R}^d} g(x_1, x_2) \check{T}_t u(t, x_1, x_2) dx_1 dx_2 \\ &\qquad = \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} g(x_1, x_2) \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} u(t, x_1, y_1) u(t, x_2, y_2) u(t, y_1, y_2) dy_1 dy_2 dx_1 dx_2 \\ &\qquad = \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} g(x_1, x_2) \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} u(t, x_1, y_1) u(t, y_1, y_2) u(t, y_2, x_2) dy_1 dy_2 dx_1 dx_2 \\ &\qquad = \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} g(x_1, x_2) u(3t, x_1, x_2) dx_1 dx_2, \end{aligned}$$

completing the proof of (3). Since for each fixed $t > 0$ we have $u(t, x, y) = u(t, y, x)$, for almost all $(x, y) \in \mathbf{R}^d \times \mathbf{R}^d$ and $u(t, x, y)$ is jointly continuous on $\mathbf{R}^d \times \mathbf{R}^d$, we obtain (2) for all $x, y \in \mathbf{R}^d$, $t > 0$.

For the proof of the positivity of the kernel $u(t, x, y)$ consider a ball B and note that by Lemma 3.1 we have that $u(t/2, x, z)$ and $u(t/2, y, z)$ are positive for all $x, y \in \mathbf{R}^d$ and for almost all $z \in B$. Thus

$$u(t, x, y) \ge \int_B u(t/2, x, z) u(t/2, z, y) dz > 0,$$

which gives (4).

Finally we show property (5). For every bounded function f on $\mathbf{R}^d$ and all $x \in \mathbf{R}^d$, $s, t \in \mathbf{R}$, $s < t$, we have

$$\begin{aligned} \int_{\mathbf{R}^d} dy\, u(t - s, x, y) f(y) &= \mathbf{E}^x \left[e^{-\int_0^{t-s} V(X_r) dr} f(X_{t-s}) \right] = \mathbf{E}^{s,x} \left[e^{-\int_s^t V(X_r) dr} f(X_t) \right] \\ &= \int_{\mathbf{R}^d} dy f(y) \mathbf{E}_{t,y}^{s,x} \left[e^{-\int_s^t V(X_r) dr} \right] p(t - s, y - x) \\ &= \int_{\mathbf{R}^d} dy f(y) \int e^{-\int_s^t V(\omega_r) dr} d\nu_{[s,t)}^{x,y}(\omega), \end{aligned}$$

which completes the proof of the theorem. □

3.3. Potential theory of fractional Schrödinger operators

Here we introduce some potential theoretic tools for fractional Schrödinger operators needed for our purposes below, and discuss some facts to be used in formulating and proving our results concerning intrinsic ultracontractivity and ground state estimates. For background we refer to [11, 12, 18, 19, 21].

The *potential operator* for the semigroup $\{T_t : t \geq 0\}$ is defined by

$$G^V f(x) = \int_0^\infty T_t f(x) dt = \mathbf{E}^x \left[\int_0^\infty e_V(t) f(X_t) dt \right],$$

for non-negative Borel functions f on $\mathbf{R}^d$. If $\int_0^\infty \|T_t\|_\infty dt < \infty$, then it follows from the proof of Lemma 3.1 that G^V is a bounded operator on $L^p(\mathbf{R}^d)$, $1 \leq p \leq \infty$. In particular, $G^V \mathbf{1} \in L^\infty$ and G^V has a symmetric kernel given by $G^V(x,y) = \int_0^\infty u(t,x,y)dt$, i.e., $G^V f(x) = \int_{\mathbf{R}^d} G^V(x,y) f(y) dy$.

The V*-Green operator* for an open set D is defined by

$$G_D^V f(x) = \int_0^\infty \mathbf{E}^x \left[t < \tau_D; e_V(t) f(X_t) \right] dt = \mathbf{E}^x \left[\int_0^{\tau_D} e_V(t) f(X_t) dt \right],$$

for non-negative Borel functions f on D. Denote

$$v_D(x) = G_D^V \mathbf{1}(x).$$

The following technical lemma will be used below.

Lemma 3.2. *Let $V \in \mathcal{K}_{\mathrm{loc}}^\alpha$ and $D \subset \mathbf{R}^d$ be a non-empty, bounded open set such that V is strictly positive and bounded on D. Then for all $x \in D$ we have*

$$\left(1 - \exp(-\sup_{y \in D} V(y))\right) \frac{\mathbf{P}^x(\tau_D > 1)}{\sup_{y \in D} V(y)} \leq v_D(x) \leq \frac{1}{\inf_{y \in D} V(y)}.$$

Proof. Fix $D \subset \mathbf{R}^d$. To simplify the notation denote $\beta = \sup_{y \in D} V(y)$ and $\zeta = \inf_{y \in D} V(y)$. For $x \in D$ we have

$$\begin{aligned} v_D(x) &= \mathbf{E}^x \left[\int_0^{\tau_D} e^{-\int_0^t V(X_s) ds} dt \right] \geq \mathbf{E}^x \left[\int_0^{\tau_D} e^{-\beta t} dt \right] \\ &= \frac{\mathbf{E}^x \left[1 - e^{-\beta \tau_D} \right]}{\beta} \geq \left(1 - e^{-\beta}\right) \frac{\mathbf{P}^x(\tau_D > 1)}{\beta}. \end{aligned}$$

Moreover,

$$v_D(x) = \mathbf{E}^x \left[\int_0^{\tau_D} e^{-\int_0^t V(X_s) ds} dt \right] \leq \mathbf{E}^x \left[\int_0^{\tau_D} e^{-\zeta t} dt \right] = \mathbf{E}^x \left[1 - e^{-\zeta \tau_D} \right] \zeta^{-1} \leq \zeta^{-1}.$$

□

Furthermore, if D' is an open set such that $D \subset D' \subseteq \mathbf{R}^d$ and f is a non-negative Borel function on D', then by the strong Markov property of stable processes we have for every $x \in D$

$$\begin{aligned} G_{D'}^V f(x) &= \mathbf{E}^x \left[\int_0^{\tau_D} e_V(t) f(X_t) dt \right] + \mathbf{E}^x \left[\int_{\tau_D}^{\tau_{D'}} e_V(t) f(X_t) dt \right] \\ &= G_D^V f(x) + \mathbf{E}^x \left[e^{-\int_0^{\tau_D} V(X_s) ds} \int_{\tau_D}^{\tau_{D'}} e^{-\int_{\tau_D}^t V(X_s) ds} f(X_t) dt \right] \\ &= G_D^V f(x) + \mathbf{E}^x \left[e_V(\tau_D) \mathbf{E}^{X_{\tau_D}} \left[\int_0^{\tau_{D'}} e_V(t) f(X_t) dt \right] \right] \\ &= G_D^V f(x) + \mathbf{E}^x \left[e_V(\tau_D) G_{D'}^V f(X_{\tau_D}) \right]. \end{aligned} \tag{3.8}$$

Define $\Phi(t) = \sup_{x\in\mathbf{R}^d} \mathbf{E}^x[t < \tau_D; e_V(t)]$, $t > 0$. If $\Phi \in L^1(0,\infty)$, then by standard arguments $G_D^V \mathbf{1} \in L^\infty$ and G_D^V is given by a symmetric kernel $G_D^V(x,y)$, $x, y \in D$, i.e., $G_D^V f(x) = \int_D G_D^V(x,y) f(y)dy$ (see [21, cor. Th. 3.18] and [11, page 58]). It is easily checked that the condition $\Phi \in L^1(0,\infty)$ is satisfied when, for instance, $V \in \mathcal{K}_{\mathrm{loc}}^\alpha$, $V \geq C_V > 0$ on D. In this case

$$\left\|G_D^V\right\|_\infty \leq \int_0^\infty \Phi(t)dt \leq C_V^{-1} < \infty.$$

The function $G_D^V(x,y)$ is called *V-Green function* of the set D.

We say that a Borel function f on $\mathbf{R}^d$ is *V-harmonic* in an open set $D \subset \mathbf{R}^d$ if

$$f(x) = \mathbf{E}^x \left[e_V(\tau_U) f(X_{\tau_U})\right], \quad x \in U, \tag{3.9}$$

for every bounded open set U with $\overline{U}$ contained in D. It is called *regular V-harmonic* in D if (3.9) holds for $U = D$. It is known [11, page 83] that every regular V-harmonic function in D is V-harmonic in D. If D is unbounded, then by convention we understand that in (3.9) $\mathbf{E}^x \left[e_V(\tau_D) f(X_{\tau_D})\right] = \mathbf{E}^x \left[\tau_D < \infty; e_V(\tau_D) f(X_{\tau_D})\right]$ holds. We always assume that the expectation in (3.9) is absolutely convergent.

For an open set $D \subset \mathbf{R}^d$ the *gauge function* is defined by $u_D(x) = \mathbf{E}^x[e_V(\tau_D)]$, $x \in D$ (see e.g. [11, p. 58], [19, 21]). When the gauge function is bounded in D, then (D,V) is said to be *gaugeable*. It is easy to check that if $V \geq 0$ on D, then gaugebility holds. If D is a bounded domain with the exterior cone property and (D,V) is gaugeable, then for $f \geq 0$ we have

$$\mathbf{E}^x[e_V(\tau_D) f(X_{\tau_D})] = \mathcal{A} \int_D G_D^V(x,y) \int_{D^c} \frac{f(z)}{|z-y|^{d+\alpha}} dz dy, \quad x \in D, \tag{3.10}$$

see [11, eq. (17), Th. 4.10].

The following estimate will be useful below. For any $\gamma \geq 0$, $\gamma \neq d$, there exists $C_\gamma > 0$ such that

$$\int_{B(x,|x|/4)^c} (1+|y|)^{-\gamma} |x-y|^{-d-\alpha} dy \leq C_\gamma |x|^{-\gamma'} \tag{3.11}$$

for $|x| \geq 1$, where $\gamma' = \min(\gamma+\alpha, d+\alpha)$. The result follows from [42, Lemma 4] for $\gamma > 0$, while for $\gamma = 0$ it is trivial.

The next lemma is a generalization to Kato class of [37, Lemma 6], where the result was obtained for $V \in L_{\mathrm{loc}}^\infty$. It concerns the comparability of functions u_D (the gauge function) and v_D in the case when D is a ball, and plays a crucial role in the proofs of the main theorems in this section.

Lemma 3.3. *Let $V \in \mathcal{K}_{\mathrm{loc}}^\alpha$, $D = B(x,r)$, $r > 0$ and $0 < \kappa < 1$. There exists a constant $C_{r,\kappa} > 0$ such that if $V \geq 0$ on D, then*

$$C_{r,\kappa}^{-1} v_D(y) \leq u_D(y) \leq C_{r,\kappa} v_D(y) \tag{3.12}$$

for all $y \in B(x,\kappa r)$, $x \in \mathbf{R}^d$.

Proof. Fix $0 < \kappa < 1$. Let $f \in C^2(\mathbf{R}^d)$ be a function such that $f \equiv 1$ on $B(x,\kappa r)$, $f \equiv 0$ on $B(x,r)^c$ and $0 \leq f \leq 1$. By [11, Prop. 3.16], we have for $z \in D$

$$G_D^V \left((-\Delta)^{\alpha/2} f + Vf\right)(z) = f(z).$$

Note that in the cited reference $e_V(t)$ is defined without a minus sign in front.

For $z \in B(x, \kappa r)$ it follows that

$$\begin{aligned} \int_D G_D^V(z,y)(-\Delta)^{\alpha/2} f(y)dy &= f(z) - \int_D G_D^V(z,y)V(y)f(y)dy \\ &\geq 1 - \int_D G_D^V(z,y)V(y)dy \\ &= 1 - \mathbf{E}^z\left[\int_0^{\tau_D} e_V(t)V(X_t)dt\right]. \end{aligned}$$

Write $V_D = V\mathbf{1}_D$. Since $V \in \mathcal{K}^\alpha_{\rm loc}$, we have $V_D \in \mathcal{K}^\alpha$. The boundedness condition (3.5) gives that for all $z \in \mathbf{R}^d$ the function $\Phi(t) = V_D(X_t)$ is $\mathbf{P}^z$-a.s. locally integrable in $(0, \infty)$ and $e_{V_D}(t)$ is $\mathbf{P}^z$-a.s. locally absolutely continuous in $(0, \infty)$. Then by using the fact that $V = V_D$ on D,

$$\int_0^{\tau_D} e_V(t)V(X_t)dt = 1 - e_V(\tau_D), \quad \mathbf{P}^z - \text{a.s.}, \ z \in \mathbf{R}^d.$$

Hence

$$u_D(z) = \mathbf{E}^z[e_V(\tau_D)] \leq \int_D G_D^V(z,y)(-\Delta)^{\alpha/2} f(y)dy \leq \left\|(-\Delta)^{\alpha/2} f\right\|_\infty v_D(z),$$

for $z \in B(x, \kappa r)$. Since $f \in C_c^2(\mathbf{R}^d)$, we have $\left\|(-\Delta)^{\alpha/2} f\right\|_\infty < \infty$. On the other hand, by (3.10), for any $z \in B(x, r)$, we have

$$\begin{aligned} u_D(z) &= \mathcal{A}\int_D G_D^V(z,y)\int_{D^c} \frac{dwdy}{|w-y|^{d+\alpha}} \\ &\geq \mathcal{A}\int_D G_D^V(z,y)\int_{B(x,2r)^c} \frac{dwdy}{|w-y|^{d+\alpha}} \\ &\geq C_r\int_D G_D^V(z,y)dy\int_{B(x,2r)^c} \frac{dw}{|w-x|^{d+\alpha}} = \frac{C_r}{r^\alpha} v_D(z). \end{aligned}$$

□

Theorem 3.4 (Uniform Harnack-type inequality). *Let $V \in \mathcal{K}^\alpha_{\rm loc}$, $r > 0$, $x_0 \in \mathbf{R}^d$ and $D = B(x_0, r)$. There exists a constant $C > 0$ such that if $V \geq 0$ on D and $f(x) = \mathbf{E}^x[e_V(\tau_D)f(X_{\tau_D})]$, for $x \in D$, $f \geq 0$, then*

$$C^{-1} v_D(x)\int_{B(x_0,r/2)^c} \frac{f(y)}{|y-x_0|^{d+\alpha}}dy \leq f(x) \leq C v_D(x)\int_{B(x_0,r/2)^c} \frac{f(y)}{|y-x_0|^{d+\alpha}}dy, \tag{3.13}$$

for every $x \in B(x_0, r/2)$.

Proof. [37, Th. 6]. □

Under the assumptions of Theorem 3.4 we obtain the following corollary. It will be a crucial step in the proof of the characterization of IUC given in Theorem 5.1 below.

Corollary 3.1. *Let $V \in \mathcal{K}^\alpha_{\rm loc}$. Suppose that there is $R > 0$ such that $V(x) \geq 1$ for $|x| \geq R$. Then there exists a constant $C > 0$ such that if $r > 0$, $x_0 \in \mathbf{R}^d$, $|x_0| - r \geq R$ and $f(x) = \mathbf{E}^x[e_V(\tau_{B(x_0,r)})f(X_{\tau_{B(x_0,r)}})]$ for $x \in B(x_0, r)$, $f \geq 0$, then*

$$f(x) \leq C\int_{B(x_0,r/2)^c} \frac{f(y)}{|y-x_0|^{d+\alpha}}dy \tag{3.14}$$

for $x \in B(x_0, r/2)$.

Proof. By the condition $|x_0| - r \geq R$ we have that $V \geq 1$ on $B(x_0, r)$. The claimed inequality is a consequence of (3.13) and the estimate

$$\mathbf{E}^x \left[\int_0^{\tau_{B(x_0,r)}} e^{-\int_0^t V(X_s)ds} dt \right] \leq \mathbf{E}^x \left[\int_0^{\tau_{B(x_0,r)}} e^{-t} dt \right] \leq \int_0^\infty e^{-t} dt = 1.$$

□

4. Ground state estimates for fractional Schrödinger operators

4.1. Ground state

The following is a standing assumption for the remainder of this paper.

Assumption 4.1. Let $\lambda_0 := \inf \operatorname{Spec} H_\alpha$ be an isolated eigenvalue. We assume that the corresponding eigenfunction φ_0 such that $\|\varphi_0\|_2 = 1$, called *ground state*, exists.

Note that when the above assumption holds, λ_0 has multiplicity one, and the ground state φ_0 is unique and has a strictly positive version with respect to Lebesgue measure. By similar arguments as in the proof of Lemma 3.3 (3) we can show that $T_t(L^\infty(\mathbf{R}^d)) \subset C_b(\mathbf{R}^d)$. Since $T_t\varphi_0(x) = \int_{\mathbf{R}^d} u(t,x,y)\varphi_0(x)dx = e^{-\lambda_0 t}\varphi_0(x)$ and the operator $T_t : L^2(\mathbf{R}^d) \to L^\infty(\mathbf{R}^d)$ is bounded, φ_0 is a continuous and bounded function. We denote the spectral gap of the operator H_α by $\Lambda := \inf(\operatorname{Spec} H_\alpha \setminus \{\lambda_0\}) - \lambda_0$.

Remark 4.1. There are few results in the literature on the existence of ground states for fractional Schrödinger operators. In [17, Th. V.1] the case of shallow potentials has been discussed. Specifically, it is shown that whenever V is non-positive, not identically zero and bounded with compact support, then H_α has a ground state φ_0 corresponding to the negative eigenvalue λ_0 if and only if $(X_t)_{t\geq 0}$ is recurrent (pointwise recurrent), i.e., if $d = 1$ and $\alpha \in (0, 1)$.

We start by a technical lemma which will be used below.

Lemma 4.1. *For all* $t > 2$

$$\sup_{x,y\in\mathbf{R}^d} |u(t,x,y) - e^{-\lambda_0 t}\varphi_0(x)\varphi_0(y)| \leq C_V e^{-\Lambda t}.$$

Proof. Notice that the function $u(t,x,y) - e^{-\lambda_0 t}\varphi_0(x)\varphi_0(y)$ is the integral kernel of the operator $T_t - e^{-\lambda_0 t}P_{\varphi_0}$, where $P_{\varphi_0} : L^2(\mathbf{R}^d) \to L^2(\mathbf{R}^d)$ is the projection onto the one-dimensional subspace of $L^2(\mathbf{R}^d)$ spanned by φ_0. Since $T_t P_{\varphi_0} = P_{\varphi_0} T_t = e^{-\lambda_0 t} P_{\varphi_0}$ for all $t > 2$, we have

$$\begin{aligned} \sup_{x,y\in\mathbf{R}^d} |u(t,x,y) - e^{-\lambda_0 t}\varphi_0(x)\varphi_0(y)| &= \sup_{\|g\|_1=1} \left\| (T_t - e^{-\lambda_0 t}P_{\varphi_0})g \right\|_\infty \\ &= \left\| T_t - e^{-\lambda_0 t}P_{\varphi_0} \right\|_{1,\infty} = \left\| T_1(T_{t-2} - e^{-\lambda_0(t-2)}P_{\varphi_0})T_1 \right\|_{1,\infty} \\ &\leq \|T_1\|_{2,\infty} \left\| T_{t-2} - e^{-\lambda_0(t-2)}P_{\varphi_0} \right\|_2 \|T_1\|_{1,2}. \end{aligned}$$

By (2) of Lemma 3.1 we obtain

$$\sup_{x,y\in\mathbf{R}^d} |u(t,x,y) - e^{-\lambda_0 t}\varphi_0(x)\varphi_0(y)| \leq C_V \left\| T_{t-2} - e^{-\lambda_0(t-2)}P_{\varphi_0} \right\|_2.$$

Finally, by the spectral theorem we have

$$\left\| T_t - e^{-\lambda_0 t} P_{\varphi_0} \right\|_2 \le e^{-\Lambda t},$$

for all $t > 0$, which completes the proof. □

4.2. **Compactness of T_t**

When for every $t > 0$ the operators T_t are compact, the spectrum of T_t is discrete consisting of eigenvalues $\lambda_0 < \lambda_1 \le \lambda_2 \le \ldots \to \infty$. The corresponding eigenfunctions φ_n satisfy $T_t \varphi_n = e^{-\lambda_n t}\varphi_n$. All φ_n are bounded continuous functions, and each λ_n has finite multiplicity. Whenever V is non-negative, $\lambda_0 > 0$, however, if V has no definite sign, then it may happen that $\lambda_0 \le 0$. In what follows we will consider this more general case.

Lemma 4.2. *Let V be a Kato-decomposable potential. If $V(x) \to \infty$ as $|x| \to \infty$, then for all $t > 0$ the operators T_t are compact.*

Proof. For any $x \in \mathbf{R}^d$ denote $D := B(x, 1)$. Let $t > 0$ be fixed. We have

$$\begin{aligned}
T_t \mathbf{1}(x) &= \mathbf{E}^x[e_V(t)] \\
&= \mathbf{E}^x\left[\tau_D \ge t; e^{-\int_0^t V(X_s)ds}\right] + \mathbf{E}^x\left[\tau_D < t; e^{-\int_0^t (V_+(X_s) - V_-(X_s))ds}\right] \\
&\le e^{-t \inf_{y\in D} V(y)} + \mathbf{E}^x\left[e^{-\int_0^{\tau_D} V_+(X_s)ds} e^{\int_0^t V_-(X_s)ds}\right] \\
&\le e^{-t \inf_{y\in D} V(y)} + \left(\mathbf{E}^x\left[e^{-\int_0^{\tau_D} 2V_+(X_s)ds}\right]\right)^{1/2} \left(\mathbf{E}^x\left[e^{\int_0^t 2V_-(X_s)ds}\right]\right)^{1/2} \\
&\le e^{-t \inf_{y\in D} V(y)} + C_{V,t}\left(\mathbf{E}^0\left[e^{-2\inf_{y\in D} V(y)\tau_{B(0,1)}}\right]\right)^{1/2}
\end{aligned}$$

by Schwarz inequality. Since $V(x) \to \infty$ as $|x| \to \infty$, $\lim_{|x|\to\infty} T_t \mathbf{1}(x) = 0$ follows.

Let now $(V_{r,t})$, $r > 0$, be the family of operators given by the kernels $v_r(t, x, y) = u(t, x, y)\mathbf{1}_{B(0,r)}(y)$, i.e., $V_{r,t} f(x) = \int_{\mathbf{R}^d} v_r(t, x, y) f(y)dy$, $f \in L^2(\mathbf{R}^d)$. We have

$$\begin{aligned}
\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} (v_r(t, x, y))^2 dxdy &= \int_{B(0,r)}\int_{\mathbf{R}^d} (u(t, x, y))^2 dxdy \\
&\le C_{V,t} \int_{B(0,r)} T_t \mathbf{1}(y)dy \\
&\le C_{V,t} e^{C_V^{(0)} + C_V^{(1)} t} |B(0, r)| < \infty.
\end{aligned}$$

Hence $V_{r,t}$ is a Hilbert-Schmidt operator, thus compact. Furthermore, by Schwarz inequality

$$\begin{aligned}
\|T_t f - V_{r,t} f\|_2^2 &= \int_{\mathbf{R}^d} \left| \int_{B(0,r)^c} u(t, x, y) f(y)dy \right|^2 dx \\
&\le \int_{\mathbf{R}^d} \int_{B(0,r)^c} u(t, x, y)dy \int_{B(0,r)^c} u(t, x, y)\, |f(y)|^2\, dydx \\
&\le e^{C_V^{(0)} + C_V^{(1)} t} \int_{\mathbf{R}^d} \int_{B(0,r)^c} u(t, x, y)\, |f(y)|^2\, dydx \\
&= e^{C_V^{(0)} + C_V^{(1)} t} \int_{B(0,r)^c} \int_{\mathbf{R}^d} u(t, x, y)dx\, |f(y)|^2\, dy \\
&\le C_{V,t}\, \|f\|_2^2 \sup_{y\in B(0,r)^c} T_t \mathbf{1}(y).
\end{aligned}$$

Since $\lim_{|x|\to\infty} T_t\mathbf{1}(x) = 0$, it follows that T_t can be approximated by compact operators $V_{r,t}$ in operator norm. Thus T_t is compact. □

4.3. **Decay of the ground state**

Notice that the condition $V(x) \to \infty$ as $|x| \to \infty$ implies that $\operatorname{supp}(V_-)$ is a bounded set and $V = V_+ \geq 0$ on $(\operatorname{supp}(V_-))^c$. Thus we are able to make use of the results of Section 3.1 for V and $D = B(x,r)$ such that $D \cap \operatorname{supp}(V_-) = \emptyset$.

Lemma 4.3. *Let V be a Kato-decomposable potential such that $V(x) \to \infty$ as $|x| \to \infty$. Put $D = B(x,1)$. Let f be a non-negative bounded function on $\mathbf{R}^d$ with the property*

$$f(x) \leq C_V^{(1)} v_D(x) \left(\sup_{y\in B(x,|x|/2)} f(y) + \int_{B(x,|x|/2)^c} f(z)|z-x|^{-d-\alpha} dz \right)$$

for any $|x| \geq 3$ such that $D \cap \operatorname{supp}(V_-) = \emptyset$. Then

$$f(x) \leq C_V^{(2)} v_D(x) |x|^{-d-\alpha}$$

for all $|x| \geq 3$ such that $D \cap \operatorname{supp}(V_-) = \emptyset$.

Proof. Suppose that for some $\gamma \geq 0$, $\gamma \neq d$, and any $x \in \mathbf{R}^d$ we have $f(x) \leq C_\gamma(1+|x|)^{-\gamma}$. It is clearly true for $\gamma = 0$. Then, for $|x| \geq 3$ such that $D \cap \operatorname{supp}(V_-) = \emptyset$, we have

$$f(x) \leq C_{V,\gamma} v_D(x) \left(\sup_{y\in B(x,|x|/2)} f(y) + \int_{B(x,|x|/2)^c} (1+|z|)^{-\gamma}|z-x|^{-d-\alpha} dz \right). \tag{4.1}$$

Hence, by (3.11),

$$f(x) \leq C_{V,\gamma} v_D(x) \left(\sup_{y\in B(x,|x|/2)} f(y) + |x|^{-\gamma'} \right), \tag{4.2}$$

for $|x| \geq 3$, $D \cap \operatorname{supp}(V_-) = \emptyset$, with $\gamma' = \min(\gamma+\alpha, d+\alpha)$. Notice that $|x| \leq 2|y|$ for $y \in B(x, |x|/2)$. Hence

$$|x|^{\gamma'} f(x) \leq C_{V,\gamma}^{(1)} v_D(x) \left(\sup_{y\in B(x,|x|/2)} |y|^{\gamma'} f(y) + 1 \right), \tag{4.3}$$

for $|x| \geq 3$, $D \cap \operatorname{supp}(V_-) = \emptyset$. Denote $g(s) = \sup_{y\in B(0,s)} |y|^{\gamma'} f(y)$. Clearly, g is non-decreasing and

$$g(s) \leq C^{(2)} s^{\gamma'}. \tag{4.4}$$

Next we will show that $g(s)$ also is bounded. Indeed, notice that by the definition of v_D we have $v_D(x) \leq (\inf_{y\in D} V(y))^{-1}$ for sufficiently large x. Since $\lim_{|x|\to\infty} V(x) = \infty$, $v_D(x) \to 0$ as $|x| \to \infty$. Thus there exists $R \geq 3$ such that $\operatorname{supp}(V_-) \cap B(0, R-1)^c = \emptyset$ and $C_{V,\gamma}^{(1)} v_D(x) \leq 2^{-\gamma'-1}$ for $|x| \geq R$. By (4.3), we get for $R \leq |x| \leq s$

$$|x|^{\gamma'} f(x) \leq 2^{-\gamma'-1} g(2|x|) + 2^{-\gamma'-1} \leq 2^{-\gamma'-1} g(2s) + 2^{-\gamma'-1}.$$

On the other hand, for $|x| \leq R$ we have

$$|x|^{\gamma'} f(x) \leq g(R) \leq g(R) + 2^{-\gamma'-1} g(2s).$$

Hence $g(2s) \geq 2^{\gamma'+1}\left(g(s) - C^{(3)}_{V,\gamma}\right)$, whenever $s \geq R$. If $g(s) \geq C^{(3)}_{V,\gamma}$, then by induction

$$g(2^n s) \geq 2^{n(\gamma'+1)} g(s) - C^{(3)}_{V,\gamma}\left(\frac{2^{n(\gamma'+1)}-1}{1-2^{-\gamma'-1}}\right), \quad n = 1, 2, ... \tag{4.5}$$

follows. Suppose now that for some $s \geq R$ we have

$$g(s) \geq \left(1 + \frac{1}{1-2^{-\gamma'-1}}\right) C^{(3)}_{V,\gamma}.$$

By (4.4) and (4.5) we obtain

$$C^{(2)} 2^{n\gamma'} s^{\gamma'} \geq g(2^n s) \geq \left(2^{n(\gamma'+1)} + \frac{1}{1-2^{-\gamma'-1}}\right) C^{(3)}_{V,\gamma} \geq 2^{n(\gamma'+1)} C^{(3)}_{V,\gamma}, \quad n = 1, 2, ...$$

This yields a contradiction and thus $g(s)$ is bounded. Therefore

$$f(x) \leq C_{V,\gamma}(1 + |x|)^{-\gamma'}, \tag{4.6}$$

with $\gamma' = \min(\gamma + \alpha, d + \alpha)$, for all $x \in \mathbf{R}^d$.

By (4.6) we can use the estimates (4.1) with $\gamma = \gamma'$ and thus obtain (4.2) with a greater γ'. Starting from (4.2) the argument can be iterated to obtain the estimate (4.6) for γ'. As soon as in an iteration step $\gamma' = d$, compare (3.11), we put $\gamma = d - \frac{\alpha}{2}$ in the subsequent step. Clearly, after $\lfloor 2 + \frac{d}{\alpha} \rfloor$ steps $f(x) \leq C_V(1+|x|)^{-d-\alpha}$, $x \in \mathbf{R}^d$, is obtained. By (4.2) this also gives $f(x) \leq C_V v_D(x)|x|^{-d-\alpha}$ for $|x| \geq 3$ such that $D \cap \operatorname{supp}(V_-) = \emptyset$. □

For $\eta > 0$ denote $V_\eta = V + \eta$ and

$$v_{D,\eta}(x) = \mathbf{E}^x\left[\int_0^{\tau_D} e_{V_\eta}(t)dt\right].$$

This implies that $v_{D,\eta} = G_D^{V_\eta}\mathbf{1}$. Our first main result in this section is the following theorem.

Theorem 4.1 (Ground state estimates). *Let $D := B(x, 1)$ and V be a Kato-decomposable potential such that $V(x) \to \infty$ as $|x| \to \infty$. Then for every $\eta \geq 0$ such that $\eta + \lambda_0 > 0$, there exist constants $C^{(1)}_{V,\eta}$ and $C^{(2)}_{V,\eta}$ such that if $D \cap \operatorname{supp}(V_-) = \emptyset$, then*

$$\frac{C^{(1)}_{V,\eta} v_{D,\eta}(x)}{(1+|x|)^{d+\alpha}} \leq \varphi_0(x) \leq \frac{C^{(2)}_{V,\eta} v_{D,\eta}(x)}{(1+|x|)^{d+\alpha}} \tag{4.7}$$

for every $x \in \mathbf{R}^d$.

Proof. Take $\eta \geq 0$ such that $\lambda_0 + \eta > 0$ (if $\lambda_0 > 0$, we may take $\eta = 0$). Notice that on integration in the equality

$$e^{-(\lambda_0+\eta)t}\varphi_0(x) = e^{-\eta t}T_t\varphi_0(x) = \mathbf{E}^x[e_{V_\eta}(t)\varphi_0(X_t)]$$

we obtain

$$\varphi_0(x) = (\lambda_0 + \eta)G^{V_\eta}\varphi_0(x).$$

By (3.8) applied to $D' = \mathbf{R}^d$ and $f = \varphi_0$ we furthermore get

$$\varphi_0(x) = (\lambda_0 + \eta)G_D^{V_\eta}\varphi_0(x) + \mathbf{E}^x[e_{V_\eta}(\tau_D)\varphi_0(X_{\tau_D})], \quad x \in D. \tag{4.8}$$

First we prove the upper bound. Let $|x| < 3$ be such that $D \cap \operatorname{supp}(V_-) = \emptyset$. By (4.8) and (3.12) we have

$$\varphi_0(x) \leq \|\varphi_0\|_\infty ((\lambda_0 + \eta)v_{D,\eta}(x) + u_{D,\eta}(x)) \leq C_{V,\eta}v_{D,\eta}(x)(1 + |x|)^{-d-\alpha}.$$

Now let $|x| \geq 3$ be such that $D \cap \operatorname{supp}(V_-) = \emptyset$. With $r = \frac{|x|}{2}$, by (4.8) and (3.10) we have

$$\begin{aligned}
\varphi_0(x) &= (\lambda_0 + \eta) \int_D G_D^{V_\eta}(x, y)\varphi_0(y)dy + \mathbf{E}^x[X_{\tau_D} \in D^c \cap B(x, r); e_{V_\eta}(\tau_D)\varphi_0(X_{\tau_D})] \\
&\quad + \mathbf{E}^x[X_{\tau_D} \in B(x, r)^c; e_{V_\eta}(\tau_D)\varphi_0(X_{\tau_D})] \\
&\leq (\lambda_0 + \eta)v_{D,\eta}(x) \sup_{y \in B(x,r)} \varphi_0(y) + u_{D,\eta}(x) \sup_{y \in B(x,r)} \varphi_0(y) \\
&\quad + \mathcal{A} \int_D G_D^{V_\eta}(x, y) \int_{B(x,r)^c} \varphi_0(z)|z - y|^{-d-\alpha} dz dy.
\end{aligned}$$

By (3.12) furthermore

$$\begin{aligned}
\varphi_0(x) &\leq (\lambda_0 + \eta)v_{D,\eta}(x) \sup_{y \in B(x,r)} \varphi_0(y) + C v_{D,\eta}(x) \sup_{y \in B(x,r)} \varphi_0(y) \\
&\quad + C \int_D G_D^{V_\eta}(x, y)dy \int_{B(x,r)^c} \varphi_0(z)|z - x|^{-d-\alpha} dz \\
&\leq C_{V,\eta} v_{D,\eta}(x) \left(\sup_{y \in B(x,r)} \varphi_0(y) + \int_{B(x,r)^c} \varphi_0(z)|z - x|^{-d-\alpha} dz \right)
\end{aligned}$$

follows. On an application of Lemma 4.3 to $f = \varphi_0$ we obtain $\varphi_0(x) \leq C_{V,\eta} v_{D,\eta}(x)|x|^{-d-\alpha}$ for $|x| \geq 3$ and $D \cap \operatorname{supp}(V_-) = \emptyset$. This gives the claimed upper bound.

To show the lower bound we use (4.8) again. Let $|x| \leq 2$; then

$$\varphi_0(x) \geq (\eta + \lambda_0)v_{D,\eta}(x) \inf_{y \in B(0,3)} \varphi_0(y) \geq C_{V,\eta} v_{D,\eta}(x)(1 + |x|)^{-d-\alpha}.$$

Take now $|x| > 2$. By (4.8) and (3.10) we have

$$\begin{aligned}
\varphi_0(x) &\geq \mathbf{E}^x[e_{q_\eta}(\tau_D)\varphi_0(X_{\tau_D})] \\
&= C \int_D G_D^{V_\eta}(x, y) \int_{D^c} \varphi_0(z)|z - y|^{-d-\alpha} dz dy \\
&\geq C \int_D G_D^{V_\eta}(x, y) \int_{B(0,1)} \varphi_0(z)|z - y|^{-d-\alpha} dz dy \\
&\geq C_V v_{D,\eta}(x)|x|^{-d-\alpha}.
\end{aligned}$$

□

By using Lemma 3.2, we can derive sharp estimates for $v_{D,\eta}(x)$ in many cases of sufficiently regular potentials. The following corollary gives explicit two-sided bounds on the ground state for potentials subject to an extra condition.

Corollary 4.1. *Let V be a Kato-decomposable potential such that $V(x) \to \infty$ as $|x| \to \infty$. Moreover, let $A \subset \{x \in \mathbf{R}^d : B(x, 1) \cap \operatorname{supp}(V_-) = \emptyset\}$, and $M_{V,A} \geq 1$ be a constant such that for every unit ball $B \subset A$*

$$V(x) \leq M_{V,A} V(y), \quad x, y \in B. \tag{4.9}$$

Then there exist constants $C^{(1)}_{V,A}$ and $C^{(2)}_{V,A}$ such that for all $x \in A$ the estimates

$$\frac{C^{(1)}_{V,A}}{V(x)(1 + |x|)^{d+\alpha}} \leq \varphi_0(x) \leq \frac{C^{(2)}_{V,A}}{V(x)(1 + |x|)^{d+\alpha}} \tag{4.10}$$

hold.

Proof. First we fix η in Theorem 4.1. If $\lambda_1 > 0$ put $\eta = 0$, if $\lambda_1 < 0$ put $\eta = -2\lambda_1$. If $\lambda_1 = 0$, then we simply choose $\eta = 1$. Fix now $x \in A$. Let $D := B(x,1)$ and $M = M_{V,A}$. Observe that by condition (4.9) we have

$$M^{-1}\eta \leq M^{-1}(V(x)+\eta) \leq \inf_{y\in D} V(y)+\eta \leq \sup_{y \in D} V(y) + \eta \leq M(V(x)+\eta).$$

This and Lemma 3.2 give

$$\frac{M'}{V(x)+\eta} \leq v_{D,\eta}(x) \leq \frac{M}{V(x)+\eta},$$

with $M' = M^{-1}\left(1-e^{-M^{-1}\eta}\right)\mathbf{P}^0(\tau_{B(0,1)} > 1)$, which implies (4.10) as a consequence of Theorem 4.1. □

Example 4.1. We illustrate the above results by specific cases of V.

(1) Whenever $|x| \geq 2$, Corollary 4.9 can be used to obtain ground state estimates for either of the potentials (i) $V(x) = |x|^{2m}$, $m \in \mathbf{N}$, (ii) $V(x) = |x|^\beta \log(1+|x|)$, $\beta > 0$, (iii) $V(x) = |x|^{-\beta}e^{|x|}$, $0 \leq \beta < \alpha$. Similarly, Corollary 4.9 directly applies for $V(x) = e^{\beta|x|}$, $\beta > 0$, for all $x \in \mathbf{R}^d$.

(2) Let $V(x) = \mathbf{1}_{|x|>1}\log|x| - \mathbf{1}_{|x|\leq 1}|x|^{-\beta}$, $0 \leq \beta < \alpha$. Then for $|x| \geq 3$

$$\frac{C_V^{(1)}}{\log|x||x|^{d+\alpha}} \leq \varphi_0(x) \leq \frac{C_V^{(2)}}{\log|x||x|^{d+\alpha}}.$$

(3) By taking $\alpha = 1$ and $m = 1$ in Example 1(i) we obtain the massless relativistic harmonic oscillator. In the case $d = 1$ the spectral properties of the operator $\sqrt{-d^2/dx^2 + x^2}$ are studied in great detail in [44]. In particular, for the large x asymptotics of the ground state

$$\varphi_0(x) = \sqrt{\frac{2}{-a_1'}}\left(\frac{p_3(a_1')}{x^4} - \frac{p_5(a_1')}{x^6} + \ldots + (-1)^N \frac{p_{2N-1}(a_1')}{x^{2N}}\right) + O\left(\frac{1}{x^{2(N+1)}}\right)$$

is found, where $a_1' \simeq -3.2482$ denotes the first zero of the derivative of the Airy function $\mathrm{Ai}(x)$, and p_n, q_n are nth order polynomials defined by the recursive relations $p_{n+1}(x) = p_n'(x) + xq_n(x)$ and $q_{n+1}(x) = p_n(x) + q_n'(x)$, with $p_0(x) \equiv 1$, $q_0(x) \equiv 0$.

Our next result concerns purely negative potentials.

Theorem 4.2. *Let V be a Kato-decomposable potential such that $V_+ \equiv 0$ and $V_-(x) \to 0$ as $|x| \to \infty$. Suppose that $\lambda_0 = \inf \mathrm{Spec}(H_\alpha) < 0$ is an isolated eigenvalue. Then for all $x \in \mathbf{R}^d$*

$$\varphi_0(x) \geq \frac{C_V}{(1+|x|)^{d+\alpha}}.$$

Proof. Similarly as before, by integrating in the equality

$$e^{-|\lambda_0|t}\varphi_0(x) = e^{-2|\lambda_0|t}T_t\varphi_0(x) = \mathbf{E}^x[e_{V_{|2\lambda_0|}}(t)\varphi_0(X_t)],$$

we obtain

$$\varphi_1(x) = |\lambda_0| G^{V_{|2\lambda_0|}}\varphi_0(x).$$

By (3.8) applied to $D' = \mathbf{R}^d$ and $f = \varphi_0$ we furthermore get

$$\varphi_0(x) = |\lambda_0| G_D^{V_{|2\lambda_0|}}\varphi_0(x) + \mathbf{E}^x[e_{V_{|2\lambda_0|}}(\tau_D)\varphi_0(X_{\tau_D})], \quad x \in D.$$

Let $|x| \leq 2$; then

$$\varphi_0(x) \geq |\lambda_0| v_{D,2|\lambda_0|}(x) \inf_{y \in B(0,3)} \varphi_0(y) \geq C_V v_{D,2|\lambda_0|}(x)(1+|x|)^{-d-\alpha}.$$

Take now $|x| > 2$. By (4.8) and (3.10) we have

$$\begin{aligned}\varphi_0(x) &\geq \mathbf{E}^x[e_{V_{2|\lambda_0|}}(\tau_D)\varphi_2(X_{\tau_D})] \\ &= C\int_D G_D^{V_{2|\lambda_0|}}(x,y)\int_{D^c}\varphi_2(z)|z-y|^{-d-\alpha}dzdy \\ &\geq C\int_D G_D^{V_{2|\lambda_0|}}(x,y)\int_{B(0,1)}\varphi_0(z)|z-y|^{-d-\alpha}dzdy \\ &\geq C_V v_{D,2|\lambda_0|}(x)|x|^{-d-\alpha}.\end{aligned}$$

Since

$$v_{D,2|\lambda_0|}(x) = \mathbf{E}^x\left[\int_0^{\tau_D} e^{\int_0^t (V_-(X_s)-2|\lambda_0|)ds}dt\right] \geq \frac{1-\mathbf{E}^0\left[e^{-2|\lambda_0|\tau_{B(0,1)}}\right]}{2|\lambda_0|},$$

the proof is complete. □

Remark 4.2. By using a martingale argument different from ours, it is possible to show that under the same assumptions as in the theorem above φ_0 is comparable to $(1+|x|)^{-d-\alpha}$ [17, Prop. IV.1-IV.3].

Example 4.2. Let $d = 1$ and $\alpha \in (0,1)$.

(1) *Potentials with compact support:* Let $V \not\equiv 0$ be a non-positive, bounded potential such that $\operatorname{supp} V \subset [-b,b]$, where $b > 0$. Then for $x \in \mathbf{R}$

$$\frac{C_V^{(1)}}{(1+|x|)^{d+\alpha}} \leq \varphi_0(x) \leq \frac{C_V^{(2)}}{(1+|x|)^{d+\alpha}}\,. \tag{4.11}$$

(2) *Potential well:* A special case of the above is

$$V(x) = \begin{cases} -a, & x\in[-b,b] \\ 0, & x\in[-b,b]^c, \end{cases}$$

where $a, b > 0$. Clearly, in this case (4.11) holds.

Example 4.3 (Coulomb potential). A case of special interest is the semi-relativistic Coulomb potential in $d = 3$, i.e., the operator $(-\Delta+m^2)^{1/2} - \frac{C}{|x|}$. It is known that for zero particle mass $m = 0$ the operator $H_1 = \sqrt{-\Delta} - \frac{C}{|x|}$ is unbounded from below when $C > \frac{2}{\pi}$. If $C \leq \frac{2}{\pi}$, then the operator H_1 is bounded from below (in fact positive), but $\operatorname{Spec} H_1 = \operatorname{Spec}_{\text{ess}} H_1 = [0,\infty)$ and $\inf \operatorname{Spec} H_1 = 0$ is not an eigenvalue (see e.g. discussion in [22, p.499]). Furthermore, as seen in Example 3.1, the Coulomb potential $V(x) = -\frac{C}{|x|}$ does not belong to the fractional Kato-class $\mathcal{K}^1$.

5. Intrinsic ultracontractivity of fractional Feynman-Kac semigroups

5.1. Analytic description of intrinsic ultracontractivity

Intrinsic ultracontractivity has been first introduced in [24] for general semigroups of compact operators and it proved to be a strong regularity property implying a number of "nice" properties of operator semigroups and their spectral properties (see, for instance, [23, 53]). Important examples include semigroups of elliptic operators and Schrödinger semigroups either on $\mathbf{R}^d$ or on domains $D \subset \mathbf{R}^d$ with Dirichlet boundary conditions [1, 25, 23, 3]. More recently, IUC has been addressed also in the case of semigroups generated by fractional Laplacians and fractional Schrödinger operators on bounded domains, see [19, 20, 40, 37].

In this section we assume that all operators T_t are compact. Recall that if V is non-negative, then the bottom of the spectrum $\lambda_0 > 0$. Since the potentials V studied in this paper have no definite sign, it may happen that $\lambda_0 \leq 0$. Our concern here is about the following properties.

Definition 5.1 (**Intrinsically ultracontractive semigroup**). *A semigroup $\{T_t : t \geq 0\}$ is called* intrinsically ultracontractive *(IUC) if for every $t > 0$ there is a constant $C_{V,t} > 0$ such that*

$$u(t,x,y) \leq C_{V,t}\, \varphi_0(x)\varphi_0(y), \quad x,y \in \mathbf{R}^d. \tag{5.1}$$

Note that originally IUC has been defined as the property that the semigroup is a bounded operator from $L^2(\mathbf{R}^d)$ to $L^\infty(\mathbf{R}^d)$ for every $t > 0$, however, for our purposes this definition is more suitable. Also, for our purposes below we propose the following notion.

Definition 5.2 (**Asymptotically intrinsically ultracontractive semigroup**). *We call a semigroup $\{T_t : t \geq 0\}$* asymptotically intrinsically ultracontractive *(AIUC) if there exists $t_0 > 0$ such that for every $t \geq t_0$ there is a constant $C_{V,t} > 0$ for which*

$$u(t,x,y) \leq C_{V,t}\, \varphi_0(x)\varphi_0(y), \quad x,y \in \mathbf{R}^d. \tag{5.2}$$

As it will be seen in Section 5.4 below IUC is a stronger property than AIUC. Clearly, it suffices to assume that (5.2) holds for some $t_0 > 0$ as by the semigroup property it extends to all $t > t_0$.

Remark 5.1. A consequence of IUC is that a similar lower bound on the kernel also holds, i.e., for every $t > 0$ there is a constant $C^{(1)}_{V,t} > 0$ such that

$$u(t,x,y) \geq C^{(1)}_{V,t}\, \varphi_0(x)\varphi_0(y), \quad x,y \in \mathbf{R}^d. \tag{5.3}$$

It is easy to see that if $\{T_t : t \geq 0\}$ is AIUC, then there is $t_0 > 0$ such that for every $t > t_0$ there is a constant $C^{(1)}_{V,t}$ satisfying inequality (5.3). An immediate consequence of this is that if the semigroup is AIUC, then $\varphi_0 \in L^1(\mathbf{R}^d)$.

Remark 5.2. The classic result for the Feynman-Kac semigroup generated by Schrödinger operators $H = (-1/2)\Delta + V$ is the following fact. If $V(x) = |x|^\beta$, then the semigroup is IUC if and only if $\beta > 2$. Moreover, if $\beta > 2$, then $cf(x) \leq \varphi_1(x) \leq Cf(x)$, $|x| > 1$, holds with some $C, c > 0$ and where

$$f(x) = |x|^{-\beta/4+(d-1)/2} \exp(-2|x|^{1+\beta/2}/(2+\beta)).$$

For details see Cor. 4.5.5, Th. 4.5.11 and Cor. 4.5.8 in [23], also [24].

The Feynman-Kac semigroup $\{T_t : t \geq 0\}$ has the particularity that in general $T_t \mathbf{1}_{\mathbf{R}^d}(x) \neq 1$. Let

$$\widetilde{u}(t,x,y) := \frac{e^{\lambda_0 t} u(t,x,y)}{\varphi_0(x)\varphi_0(y)}. \tag{5.4}$$

Definition 5.3 (**Intrinsic fractional Feynman-Kac semigroup**). *We call the one-parameter semigroup $\{\widetilde{T}_t : t \geq 0\}$ with integral kernel $\widetilde{u}(t,x,y)$, i.e.,*

$$\widetilde{T}_t f(x) = \int_{\mathbf{R}^d} f(y)\widetilde{u}(t,x,y)\varphi_0^2(y)dy \tag{5.5}$$

intrinsic fractional Feynman-Kac semigroup.

Note that even though the L^p-norms of the operators T_t can be larger than 1, the operators $\widetilde{T}_t$ are always contractions. Moreover, the intrinsic semigroup is more natural than $\{T_t : t \geq 0\}$ since for every $t > 0$ and $x \in \mathbf{R}^d$, $\widetilde{T}_t \mathbf{1}_{\mathbf{R}^d}(x) = 1$.

The intrinsic semigroup is acting on $L^2(\mathbf{R}^d, \varphi_0^2 dx)$ and is generated by the operator

$$\widetilde{H} := U^{-1}(-(-\Delta)^{\alpha/2} - V)U$$

where the unitary map $U : L^2(\mathbf{R}^d, \varphi_0^2(x)dx) \to L^2(\mathbf{R}^d, dx)$ is defined by

$$Uf(x) = \varphi_0(x) f(x). \tag{5.6}$$

For sufficiently regular functions f (e.g., from Schwartz space) the operator $\widetilde{H}$ can be computed explicitly to be

$$\begin{aligned} \widetilde{H}f(x) \;=\; & \mathcal{A} \int_{\mathbf{R}^d} \frac{f(x+y) - f(x) - (y, \nabla f(y))_{\mathbf{R}^d}}{|y|^{d+\alpha}} \, \frac{\varphi_0(x+y)}{\varphi_0(x)} \mathbf{1}_{\{|y| \leq 1\}} dy \\ & + \mathcal{A} \int_{\mathbf{R}^d} \frac{(y, \nabla f(y))_{\mathbf{R}^d}}{|y|^{d+\alpha}} \, \frac{\varphi_0(x+y) - \varphi_0(x)}{\varphi_0(x)} \mathbf{1}_{\{|y| \leq 1\}} dy. \end{aligned} \tag{5.7}$$

Lemma 5.1. *The following two conditions are equivalent.*

(1) *The semigroup $\{T_t : t \geq 0\}$ is AIUC.*
(2) *The property*

$$\widetilde{u}(t, x, y) \stackrel{t \to \infty}{\longrightarrow} 1, \tag{5.8}$$

holds, uniformly in $(x, y) \in \mathbf{R}^d \times \mathbf{R}^d$.

Proof. The implication (2) $\Rightarrow$ (1) is immediate, we only show the converse statement. We have for every $x, y \in \mathbf{R}^d$ and $t > 2t_0$

$$\begin{aligned} & |\widetilde{u}(t, x, y) - 1| \\ & = \left| \int \int \frac{u(t_0, x, z) u(t - 2t_0, z, w) u(t_0, w, y)}{e^{-\lambda_0 t} \varphi_0(x) \varphi_0(y)} dz dw - \frac{e^{-\lambda_0 t} \varphi_0(x) \varphi_0(y)}{e^{-\lambda_0 t} \varphi_0(x) \varphi_0(y)} \right| \\ & = \left| \int \int \frac{u(t_0, x, z) \varphi_0(z) \left(u(t - 2t_0, z, w) - e^{-\lambda_0 (t - 2t_0)} \varphi_0(z) \varphi_0(w) \right) u(t_0, w, y) \varphi_0(w)}{e^{-\lambda_0 t} \varphi_0(x) \varphi_0(z) \varphi_0(w) \varphi_0(y)} dz dw \right| \\ & \leq e^{\lambda_0 t} \left\| \frac{u(t_0, x, y)}{\varphi_0(x) \varphi_0(y)} \right\|_\infty^2 \int \int \left| u(t - 2t_0, z, w) - e^{-\lambda_0 (t - 2t_0)} \varphi_0(z) \varphi_0(w) \right| \varphi_0(z) \varphi_0(w) dz dw \\ & \leq C e^{\lambda_0 t} \left(\int \int \left| u(t - 2t_0, z, w) - e^{-\lambda_0 (t - 2t_0)} \varphi_0(z) \varphi_0(w) \right|^2 dz dw \right)^{1/2}. \end{aligned}$$

The last factor on the right hand side is the Hilbert-Schmidt norm of the operator $T_{t-2t_0} - e^{-\lambda_0 (t - 2t_0)} P_{\varphi_0}$, where $P_{\varphi_0} : L^2(\mathbf{R}^d) \to L^2(\mathbf{R}^d)$ is the projection onto the one dimensional subspace of $L^2(\mathbf{R}^d)$ spanned by φ_0. This gives

$$|\widetilde{u}(t, x, y) - 1| \leq C e^{\lambda_0 t} \left(\sum_{k=1}^\infty e^{-2\lambda_k (t - 2t_0)} \right)^{1/2} = C e^{2t_0 \lambda_1} e^{-(\lambda_1 - \lambda_0) t} \left(\sum_{k=1}^\infty e^{-2(\lambda_k - \lambda_1)(t - 2t_0)} \right)^{1/2}.$$

By dominated convergence the last sum converges to the multiplicity of λ_1 as $t \to \infty$. Since $\lambda_1 > \lambda_0$, (5.8) follows. □

5.2. **Probabilistic description of intrinsic ultracontractivity**

In Section 6 below it will be seen that (A)IUC has a direct impact on the properties of Gibbs measures defined for stable processes. To gain information on the structure of these measures (such as typical path behaviour) we would like to understand IUC in an alternative probabilistic way. The following ergodicity property is a suggestive consequence of AIUC.

Proposition 5.1. *Let V be a Kato-decomposable potential. The probability measure ρ defined by*

$$\rho(A) = \int_A \varphi_0^2(y)dy$$

is a stationary distribution for $\{\widetilde{T}_t : t \geq 0\}$, i.e., for every $t > 0$ and Borel set $A \subset \mathbf{R}^d$

$$\int_{\mathbf{R}^d} \widetilde{T}_t \mathbf{1}_A(x)\rho(dx) = \rho(A).$$

Moreover, if the semigroup $\{T_t : t \geq 0\}$ is AIUC, then for every $t > 0$ and Borel set $A \in \mathbf{R}^d$

$$\lim_{t\to\infty} \widetilde{T}_t \mathbf{1}_A(x) = \rho(A)$$

holds, uniformly in $x \in \mathbf{R}^d$.

Proof. The convergence part of the proposition is a direct consequence of Lemma 5.1. For the proof of stationarity of the measure ρ we write

$$\begin{aligned}\int_{\mathbf{R}^d} \widetilde{T}_t \mathbf{1}_A(x)\rho(dx) &= \int_{\mathbf{R}^d}\int_A \widetilde{u}(t,x,y)\varphi_0(x)\varphi_0(y)\varphi_0^2(y)dy\varphi_0^2(x)dx \\ &= \int_A\int_{\mathbf{R}^d} e^{\lambda_0 t}u(t,x,y)\varphi_0(x)dx\varphi_0(y)dy = \int_A \varphi_0^2(y)dy.\end{aligned}$$

□

Although this shows that an implication of AIUC is uniform convergence of the intrinsic semigroup $\{\widetilde{T}_t : t \geq 0\}$ to the stationary distribution of density φ_0^2, it is useful to understand in a probabilistic way what IUC and AIUC mean on the level of the semigroup $\{T_t : t \geq 0\}$.

For the remainder of this section we will use the following conditions.

Assumption 5.1. Suppose that V is a Kato-decomposable potential such that for every $t > 0$ operators T_t are compact. Moreover, let

$$T_t \mathbf{1}_{\mathbf{R}^d}(x) \leq C_{D,t} T_t \mathbf{1}_D(x)\,, \tag{5.9}$$

where $t > 0$, $x \in \mathbf{R}^d$, D is a bounded non-empty Borel subset of $\mathbf{R}^d$ and $C_{D,t} > 0$. We will consider the following assumptions.

(1) For every $t > 0$ there exists D and $C_{D,t}$ such that (5.9) holds for all $x \in \mathbf{R}^d$.
(2) For every $t > 0$ and D there exists $C_{D,t}$ such that (5.9) holds for all $x \in \mathbf{R}^d$.
(3) There exists $t_0 > 0$ such that for every $t > t_0$ there is D and $C_{D,t}$ such that (5.9) holds for all $x \in \mathbf{R}^d$.
(4) There exists $t_0 > 0$ such that for every $t > 0$ and every D there is $C_{D,t}$ such that (5.9) holds for all $x \in \mathbf{R}^d$.

Clearly, by the semigroup property $T_t T_s = T_{t+s}$ whenever (5.9) holds for some $t > 0$, set D and constant $C_{D,t}$, then it holds for all $s \geq t$ with the same D and $C_{D,t}$.

First we note that IUC can be characterized by the above conditions.

Lemma 5.2. *Let Assumption 5.1 (1) hold. Then the semigroup $\{T_t : t \geq 0\}$ is IUC. Let the semigroup $\{T_t : t \geq 0\}$ be IUC. Then Assumption 5.1 (2) holds.*

Proof. First assume that the semigroup $\{T_t : t \geq 0\}$ is IUC. Fix $t > 0$ and a bounded set $D \subset \mathbf{R}^d$. For $x \in \mathbf{R}^d$ we have

$$T_t \mathbf{1}_{\mathbf{R}^d}(x) = \int_{\mathbf{R}^d} u(t,x,y)dy \leq C_{V,t} \|\varphi_0\|_1 \varphi_0(x).$$

On the other hand,

$$T_t \mathbf{1}_D(x) = \int_D u(t,x,y)dy \geq C_{V,t}\varphi_0(x) \int_D \varphi_0(y)dy$$

and Assumption 5.1 (2) follows.

Let now Assumption 5.1 (1) be satisfied. For every $x, y \in \mathbf{R}^d$ and $t > 0$ by the semigroup property

$$\begin{aligned} u(t,x,y) &= \int_{\mathbf{R}^d}\int_{\mathbf{R}^d} u(t/3,x,z)u(t/3,z,w)u(t/3,w,y)dzdw \\ &\leq C_{V,t} T_{t/3}\mathbf{1}_{\mathbf{R}^d}(x) T_{t/3}\mathbf{1}_{\mathbf{R}^d}(y) \leq C_{V,t} T_{t/3}\mathbf{1}_D(x) T_{t/3}\mathbf{1}_D(y) \\ &\leq \frac{C_{V,t}}{(\inf_{y\in D}\varphi_0(y))^2} T_{t/3}\varphi_0(x) T_{t/3}\varphi_0(y) = C_{V,t} e^{-2\lambda_0 t/3}\varphi_0(x)\varphi_0(y). \end{aligned}$$

□

A straightforward corollary of the above lemma is the following.

Corollary 5.1. *Consider the semigroup $\{T_t : t \geq 0\}$.*

1. *If Assumption 5.1 (3) holds, then $\{T_t : t \geq 0\}$ is AIUC. If $\{T_t : t \geq 0\}$ is AIUC, then Assumption 5.1 (4) holds.*
2. *The semigroup $\{T_t : t \geq 0\}$ is IUC if and only if either of the two equivalent Assumptions 5.1 (1) and 5.1 (2) is satisfied, and it is AIUC if and only if either of the two equivalent Assumptions 5.1 (3) and 5.1 (4) holds.*

Using the above statements we can give an equivalent probabilistic definition of IUC and AIUC.

Definition 5.4. *Let V be a Kato-decomposable potential. We say that the corresponding semigroup $\{T_t : t \geq 0\}$ is* intrinsically ultracontractive (IUC) *whenever for every $t > 0$ there exist a non-empty bounded Borel set $D \subset \mathbf{R}^d$ and a constant $C_{V,t} > 0$ such that for all $x \in \mathbf{R}^d$*

$$\mathbf{E}^x\left[X_t \in D^c; e_V(t)\right] \leq C_{V,t}\, \mathbf{E}^x\left[X_t \in D; e_V(t)\right] \tag{5.10}$$

holds. We say that the semigroup $\{T_t : t \geq 0\}$ is asymptotically intrinsically ultracontractive (AIUC) *whenever there exists $t_0 > 0$ such that for every $t \geq t_0$ there is a non-empty bounded set $D \subset \mathbf{R}^d$ and a constant $C_{V,t} > 0$ such that for all $x \in \mathbf{R}^d$ inequality* (5.10) *holds.*

We will show in the next two subsections that IUC and AIUC are properties depending only on the behaviour of the Kato-decomposable potential V at infinity. In particular, the local behaviour of the positive and negative parts, even with possible singularities, does not have an effect on IUC and AIUC.

For a pathwise interpretation of IUC we have the following heuristic description (more details and proofs are discussed in [36]). Consider first processes generated by usual Schrödinger operators. A calculation of the generator of the intrinsic semigroup then shows that when the ground state is sufficiently smooth these are Itô diffusions $(Y_t)_{t\geq 0}$ satisfying the stochastic differential equation

$$dY_t = (\nabla \ln \varphi_0(Y_t))dt + dB_t. \tag{5.11}$$

It is known that in many cases ground states decay exponentially at infinity, while for potentials growing at infinity they decay more rapidly. For an intuition, let $\varphi_0(x) = e^{-|x|^n}$ and $V(x) = |x|^{2m}$. Then by using the Schrödinger equation we roughly have

$$-\frac{1}{2}\left(n(n-1)|x|^{n-2} + n^2|x|^{2n-2}\right)\varphi_0 + |x|^{2m}\varphi_0 = \lambda_0\varphi_0.$$

Comparing in the leading order when $|x|$ is large, we expect that $n = m+1$; in fact, this is rigorously true [15]. For $m = 1$ $(Y_t)_{t\geq 0}$ is an Ornstein-Uhlenbeck process whose mean drift is exponentially decaying in t. For $m > 1$ a calculation indicates using the above estimates that the mean drift is polynomially decaying. A combination with Remark 5.2 this suggests that for Itô diffusions IUC occurs when the paths are forced to fluctuate under a strong (only polynomially decaying) drift, while IUC is lost when paths fluctuate more freely as the drift weakens to a negative exponential. Thus IUC appears as a path concentration mechanism around the bottom of the energy landscape (determined by the minima of V). By using (5.7) it can be expected that a similar mechanism applies in the case of fractional Schrödinger operators, however, here the paths will jump. Through a jump the process can save much of the energy cost spent on fluctuating against the gradient field of V, and the paths can concentrate more effectively around the bottom of the energy landscape. Therefore, IUC should hold for a larger class of potentials in the case of fractional Schrödinger operators than in the case of Schrödinger operators, as it is indeed the case by comparing the borderline cases (see Remark 5.2 and Corollary 5.2 below).

Using the Feynman-Kac semigroup it is seen that the effect of the potential on the distribution of paths is a concurrence of killing at a rate of $e^{-\int_0^t V_+(X_s)ds}$ and mass generation at a rate of $e^{\int_0^t V_-(X_s)ds}$. However, if $V(x) \to \infty$ as $|x| \to \infty$, then outside some compact set only the killing effect occurs and $\mathbf{E}^x[e^{-\int_0^t V(X_s)ds}]$ is the probability of survival of the process under the potential up to time t. The Feynman-Kac functional suggests that asymptotically the probability of survival of the process staying near the starting point x is roughly $e^{-tV(x)}$, while the probability of surviving while travelling to a region D where the killing part of the potential is smaller is $\mathbf{P}^x(X_t \in D)$. From inequality (5.10) we see that the semigroup $\{T_t : t \geq 0\}$ is IUC if and only if the probability that the process under V survives up to time t far from $\inf V$ is bounded by the probability that the process survives up to time t and is in some bounded region D, independently of its starting point. Comparing these two probabilities suggests that the outcome of the competing effects will be decided by the ratio $V(x)/|\log \mathbf{P}^x(X_t \in D)|$.

Example 5.1. Take D to be a bounded neighbourhood of the location of $\inf V$ (in the examples below, the origin) and $x \in D^c$ such that $\mathrm{dist}(x, D)$ is large. Denote in each case below by $\mathbf{P}^x$ the measure of the process with $V \equiv 0$.

(1) *Brownian motion:* The borderline case $V(x) = |x|^2$ (Remark 5.2) can also be seen probabilistically. Then

$$\mathbf{P}^x(B_t \in D) = \frac{1}{(4\pi t)^{d/2}} \int_D e^{\frac{|y-x|^2}{4t}} dy,$$

which yields

$$C_t^{(1)} e^{-C_{t,D}^{(2)}|x|^2} \leq \mathbf{P}^x(B_t \in D) \leq C_t^{(3)} e^{-C_{t,D}^{(4)}|x|^2},$$

with $C^{(1)}, ..., C^{(4)} > 0$, giving $-\log \mathbf{P}^x(X_t \in D) \asymp |x|^2$.

(2) *Symmetric stable process:* By using estimate (2.4) we derive that

$$\mathbf{P}^x(X_t \in D) = \int_D p(t, y-x)dy \asymp t\frac{1}{|x|^{d+\alpha}} = te^{-(d+\alpha)\log|x|}.$$

This gives $-\log \mathbf{P}^x(X_t \in D) \asymp \log |x|$ for the borderline case of the potential, which agrees with Theorems 5.2 and 5.3.

(3) *Relativistic stable process:* Let $(X_t^m)_{t\geq 0}$ be a process in $\mathbf{R}^d$ with parameters $\alpha \in (0,2)$, $m > 0$, generated by the Schrödinger operator $(-\Delta + m^2)^{\alpha/2} - m + V$. It is proven in [41] that in case of non-negative potentials comparable on unit balls the corresponding Schrödinger semigroup is IUC if and only if $\lim_{|x|\to\infty} \frac{V(x)}{|x|} = \infty$. A similar calculation as above using estimates on the transition density (see e.g. [52]) shows that

$$C^{(1)} e^{-C^{(2)}|x|} \leq \mathbf{P}^x(X_t^m \in D) \leq C^{(3)} e^{-C^{(4)}|x|},$$

where $C^{(1)}, ..., C^{(4)} > 0$ depend on m, t and D only, i.e., $-\log \mathbf{P}^x(X_t \in D) \asymp |x|$.

(4) In the light of these examples it is tempting to suppose that the borderline cases can be similarly identified for a wider class of time and space homogenous Markov processes [36]. Let $(B_{S_t})_{t\geq 0}$ be subordinate Brownian motion in $\mathbf{R}^d$, $d \geq 1$, with respect to a subordinator $(S_t)_{t\geq 0}$. It is known that for any given subordinator there exists a unique Bernstein function Ψ such that $\lim_{s\to 0^+} \Psi(s) = 0$ and the operator $\Psi(-\Delta)$ is the generator of the subordinate Brownian motion. Let $\{T_t : t \geq 0\}$ be the Feynman-Kac semigroup for the process $(B_{S_t})_{t\geq 0}$, and V be a Ψ-*Kato-decomposable* potential corresponding to the Schrödinger operator $\Psi(-\Delta) + V$ defined by the right hand side of the Feynman-Kac formula [33]

$$T_t f(x) = \mathbf{E}^x \left[e^{-\int_0^t V(B_{S_r})dr} f(B_{S_t}) \right].$$

Then it can be conjectured that the semigroup $\{T_t : t \geq 0\}$ is IUC if there exists a non-empty bounded open set D and $t > 0$ such that

$$\lim_{|x|\to\infty} \frac{V(x)}{|\log \mathbf{P}^x(B_{S_t} \in D)|} = \infty.$$

Remark 5.3. The following result in [38], which has become available just before submitting our paper, is close to our intuition on IUC and it adds to the understanding of the probabilistic mechanism of IUC. Let V be a non-negative Kato-class potential. The conditions below are equivalent:

(1) The semigroup $\{T_t : t \geq 0\}$ is AIUC.

(2) The semigroup $\{T_t : t \geq 0\}$ is uniformly conditionally ergodic, i.e., for every Borel set $A \subset \mathbf{R}^d$ and $x \in \mathbf{R}^d$

$$\lim_{t\to\infty} \frac{T_t \mathbf{1}_A(x)}{T_t \mathbf{1}_{\mathbf{R}^d}(x)} = \frac{\int_A \varphi_0(y)dy}{\|\varphi_0\|_1}$$

uniformly with respect to the starting point x and the set A. Moreover, the measure $\theta(A) = \|\varphi_0\|_1^{-1} \int_A \varphi_0(y)dy$ is the unique quasi-stationary distribution for $\{T_t : t \geq 0\}$, i.e., θ is the only probability measure such that for every $t > 0$ and Borel set A

$$\frac{\int_{\mathbf{R}^d} \theta(dx) T_t \mathbf{1}_A(x)}{\int_{\mathbf{R}^d} \theta(dx) T_t \mathbf{1}_{\mathbf{R}^d}(x)} = \theta(A).$$

In particular, condition (2) above provides a useful description of the process under IUC potentials V for long times t (i.e., when AIUC holds). Since by the semigroup property (1) is equivalent to the t_0-compact domination used by the authors, the result is a direct consequence of Th. 1, Rem. 2 and Cor. 2. in [38].

5.3. **Intrinsic ultracontractivity of fractional Feynman-Kac semigroups**

Our main goal here is to establish and characterize IUC for fractional Schrödinger operators with Kato-decomposable potentials. While IUC usually is defined and considered for semigroups generated by non-negative compact self-adjoint operators, we do not assume positivity and include also the case when the bottom of the spectrum may be negative.

First we need the following technical lemma.

Lemma 5.3. *Let V be a Kato-decomposable potential and $D \subset \mathbf{R}^d$ be an arbitrary open set. Then for every $t > 0$ we have that*

(1) $\mathbf{E}^x\left[\frac{t}{2} \geq \tau_D; e_q(t)\right] \leq C_{V,t}\mathbf{E}^x\left[e_V(\tau_D)T_{\frac{t}{2}}\mathbf{1}(X_{\tau_D})\right]$

(2) $\mathbf{E}^x\left[\frac{t}{2} < \tau_D; e_V(t)\right] \leq C_{V,t}\mathbf{E}^x\left[\frac{t}{4} < \tau_D; e_V\left(\frac{t}{4}\right)\right]\sup_{y\in D} T_{3t/4}\mathbf{1}(y)$.

Proof. By the usual and the strong Markov properties we obtain

$$\begin{aligned}
&\mathbf{E}^x\left[\frac{t}{2} \geq \tau_D; e_{V_+}(t)e_{-V_-}(t)\right]\\
&\leq \mathbf{E}^x\left[\frac{t}{2} \geq \tau_D; e_V(\tau_D)e^{-\int_{\tau_D}^{\frac{t}{2}+\tau_D} V_+(X_s)ds}e^{\int_{\tau_D}^{t+\tau_D} V_-(X_s)ds}\right]\\
&\leq \mathbf{E}^x\left[e_V(\tau_D)\mathbf{E}^{X_{\tau_D}}\left[e_{V_+}\left(\frac{t}{2}\right)e_{-V_-}(t)\right]\right]\\
&= \mathbf{E}^x\left[e_V(\tau_D)\mathbf{E}^{X_{\tau_D}}\left[e_{V_+}\left(\frac{t}{2}\right)e_{-V_-}\left(\frac{t}{2}\right)\mathbf{E}^{X_{t/2}}\left[e_{-V_-}\left(\frac{t}{2}\right)\right]\right]\right]\\
&\leq \sup_{y\in\mathbf{R}^d}\mathbf{E}^y\left[e_{-V_-}\left(\frac{t}{2}\right)\right]\mathbf{E}^x\left[e_V(\tau_D)\mathbf{E}^{X_{\tau_D}}\left[e_{V_+}\left(\frac{t}{2}\right)e_{-V_-}\left(\frac{t}{2}\right)\right]\right]\\
&\leq C_{V,t}\mathbf{E}^x\left[e_V(\tau_D)\mathbf{E}^{X_{\tau_D}}\left[e_V\left(\frac{t}{2}\right)\right]\right].
\end{aligned}$$

This gives (1). Similarly, once again by the Markov property

$$\begin{aligned}
\mathbf{E}^x\left[\frac{t}{2} < \tau_D; e_V(t)\right] &= \mathbf{E}^x\left[\frac{t}{4} < \tau_D; e_V\left(\frac{t}{4}\right)\mathbf{E}^{X_{t/4}}\left[\frac{t}{4} < \tau_D; e_V\left(\frac{3t}{4}\right)\right]\right]\\
&\leq \sup_{y\in D} T_{3t/4}\mathbf{1}(y)\,\mathbf{E}^x\left[\frac{t}{4} < \tau_D; e_V\left(\frac{t}{4}\right)\right],
\end{aligned}$$

which completes the proof. □

Our main result on the characterization of IUC is as follows.

Theorem 5.1 (**Characterization of IUC**). *Let V be a Kato-decomposable potential such that $V(x) \to \infty$ as $|x| \to \infty$. The following conditions are equivalent:*

(1) *The semigroup $\{T_t : t \geq 0\}$ is intrinsically ultracontractive.*

(2) *For any $t > 0$ there is a constant $C_{V,t}$ such that for all $x, y \in \mathbf{R}^d$*

$$u(t, x, y) \leq C_{V,t}(1 + |x|)^{-d-\alpha}(1 + |y|)^{-d-\alpha}.$$

(3) *For any $t > 0$ there is a constant $C_{V,t}$ such that for all $r > 0$, $x \in \overline{B}(0, r)^c$*

$$\mathbf{E}^x[t < \tau_{\overline{B}(0,r)^c}; e_V(t)] \leq C_{V,t}(1 + r)^{-d-\alpha}.$$

(4) *For any $t > 0$ there is a constant $C_{V,t}$ such that for all $x \in \mathbf{R}^d$*

$$T_t\mathbf{1}(x) \leq C_{V,t}(1 + |x|)^{-d-\alpha}.$$

Proof. We proceed in a succession of steps.

(Step 1) For the proof of the implication (1) ⇒ (2) consider the set

$$A = \left\{ x \in \mathbf{R}^d : B(x,1) \cap \operatorname{supp}(V_-) = \emptyset \right\}.$$

Clearly, by the assumption on the potential A^c is bounded and $V \geq 0$ on each $B(x,1)$ for $x \in A$. If $x, y \in A$, then (2) follows by the definition of IUC and the upper bound in Theorem 4.1. Whenever $x, y \in A^c$, then the boundedness of $u(t,x,y)$ and A^c give (2). If now $x \in A$, $y \in A^c$, then we have

$$u(t,x,y) \leq C_{V,t}\varphi_0(x)\varphi_0(y) \leq C_{V,t}\varphi_0(x) \leq C_{V,t}(1+|x|)^{-d-\alpha}(1+|y|)^{-d-\alpha}$$

by an argument similar as above. The case $x \in A^c$, $y \in A$ follows by symmetry.

(Step 2) By (2) we have

$$\begin{aligned} \mathbf{E}^x[t < \tau_{\overline{B}(0,r)^c}; e_V(t)] &\leq \mathbf{E}^x[X_t \in \overline{B}(0,r)^c; e_V(t)] \\ &= \int_{\overline{B}(0,r)^c} u(t,x,y)dy \leq C_{V,t}(1+|x|)^{-d-\alpha} \leq C_{V,t}(1+r)^{-d-\alpha}, \end{aligned}$$

for $x \in \overline{B}(0,r)^c$. This gives (3).

(Step 3) Next we prove (3) ⇒ (4). Let $R > 1$ be sufficiently large so that $V(y) \geq 1$ for $|y| \geq R$. Let $|x| \geq 2R$, $r = |x|/2$ and $D = B(x,r)$. It is clear that $D \cap \operatorname{supp}(V_-) = \emptyset$. We write

$$T_t\mathbf{1}(x) = \mathbf{E}^x\left[\frac{t}{2} < \tau_D; e_V(t)\right] + \mathbf{E}^x\left[\frac{t}{2} \geq \tau_D; e_V(t)\right].$$

By condition (3) and Lemma 5.3 we obtain

$$\begin{aligned} \mathbf{E}^x\left[\frac{t}{2} < \tau_D; e_V(t)\right] &\leq C_{V,t}\mathbf{E}^x\left[\frac{t}{4} < \tau_D; e_V\left(\frac{t}{4}\right)\right] \\ &\leq C_{V,t}\mathbf{E}^x\left[\frac{t}{4} < \tau_{\overline{B}(0,r)^c}; e_V\left(\frac{t}{4}\right)\right] \leq C_{V,t}(1+|x|)^{-d-\alpha} \end{aligned}$$

and

$$\mathbf{E}^x\left[\frac{t}{2} \geq \tau_D; e_V(t)\right] \leq C_{V,t}\mathbf{E}^x\left[e_V(\tau_D)T_{t/2}\mathbf{1}(X_{\tau_D})\right].$$

Thus

$$T_t\mathbf{1}(x) \leq C_{V,t}\left((1+|x|)^{-d-\alpha} + \mathbf{E}^x\left[e_V(\tau_D)T_{t/2}\mathbf{1}(X_{\tau_D})\right]\right). \tag{5.12}$$

We need to estimate the last expectation. Put

$$f(y) = \begin{cases} \mathbf{E}^y\left[e_V(\tau_D)T_{t/2}\mathbf{1}(X_{\tau_D})\right] & \text{for} \quad y \in D, \\ T_{t/2}\mathbf{1}(y) & \text{for} \quad y \in D^c. \end{cases}$$

Then $f(y) = \mathbf{E}^y\left[e_V(\tau_D)f(X_{\tau_D})\right]$, $y \in D$, and by (3.14) we obtain

$$\begin{aligned} f(x) &\leq C\int_{B(x,r/2)^c} \frac{f(y)}{|y-x|^{d+\alpha}}dy \\ &= C\left(\int_{D\setminus B(x,r/2)} \frac{\mathbf{E}^y\left[e_V(\tau_D)T_{t/2}\mathbf{1}(X_{\tau_D})\right]}{|y-x|^{d+\alpha}}dy + \int_{D^c} \frac{T_{t/2}\mathbf{1}(y)}{|y-x|^{d+\alpha}}dy\right). \end{aligned} \tag{5.13}$$

Hence by a combination of (5.12) and (5.13)

$$T_t\mathbf{1}(x) \le C_{V,t}\Bigg((1+|x|)^{-d-\alpha} + \int_{B(x,r/2)^c} \frac{T_{t/2}\mathbf{1}(y)}{|y-x|^{d+\alpha}}dy \tag{5.14}$$
$$+ \sup_{y\in D} \mathbf{E}^y\left[e_V(\tau_D)T_{t/2}\mathbf{1}(X_{\tau_D})\right](1+|x|)^{-\alpha}\Bigg)$$

is obtained. The fact that $V \ge 1$ on D and (3.10) imply for $y \in D$ that

$$\begin{aligned}\mathbf{E}^y\left[e_V(\tau_D)T_{t/2}\mathbf{1}(X_{\tau_D})\right] &= \mathbf{E}^y\left[X_{\tau_D} \in B(x,3r/2)\backslash D; e_V(\tau_D)T_{t/2}\mathbf{1}(X_{\tau_D})\right] \\ &\quad + \mathbf{E}^y\left[X_{\tau_D} \in B(x,3r/2)^c; e_V(\tau_D)T_{t/2}\mathbf{1}(X_{\tau_D})\right] \\ &\le u_D(y) \sup_{z\in B(x,3r/2)} T_{t/2}\mathbf{1}(z) + \mathcal{A}\int_D G_D^V(y,z)\int_{B(x,3r/2)^c}\frac{T_{t/2}\mathbf{1}(v)}{|v-z|^{d+\alpha}}dvdz \\ &\le u_D(y) \sup_{z\in B(x,3r/2)} T_{t/2}\mathbf{1}(z) + Cv_D(y)\int_{B(x,3r/2)^c}\frac{T_{t/2}\mathbf{1}(v)}{|v-x|^{d+\alpha}}dv \\ &\le \sup_{z\in B(x,3r/2)} T_{t/2}\mathbf{1}(z) + C\int_{B(x,r/2)^c}\frac{T_{t/2}\mathbf{1}(y)}{|y-x|^{d+\alpha}}dy.\end{aligned}$$

Thus we obtain from (5.14)

$$T_t\mathbf{1}(x) \le C_{V,t}\left((1+|x|)^{-d-\alpha} + \int_{B(x,r/2)^c}\frac{T_{t/2}\mathbf{1}(y)}{|y-x|^{d+\alpha}}dy + \sup_{z\in B(x,3r/2)} T_{t/2}\mathbf{1}(z)(1+|x|)^{-\alpha}\right). \tag{5.15}$$

Suppose now that for some $\gamma \ge 0$, $\gamma \ne d$, we have $T_t\mathbf{1}(x) \le C_{V,t,\gamma}(1+|x|)^{-\gamma}$, for all $x \in \mathbf{R}^d$, $t > 0$. This clearly holds for $\gamma = 0$. Then by (5.15) and (3.11) we obtain

$$T_t\mathbf{1}(x) \le C_{V,t}(1+|x|)^{-d-\alpha} + C_{V,t,\gamma}(1+|x|)^{-\gamma-\alpha} \tag{5.16}$$
$$+ C_{V,t,\gamma}\int_{B(x,r/2)^c}(1+|y|)^{-\gamma}|y-x|^{-d-\alpha}dy \le C_{V,t,\gamma}(1+|x|)^{-\gamma'}$$

for $\gamma' = \min(\gamma+\alpha, d+\alpha)$ and $|x| \ge 2R$. Also, $T_t\mathbf{1}(x) \le C_{V,t,\gamma}(1+|x|)^{-\gamma'}$ for $|x| \le 2R$.

Now, starting from (5.15) again and taking $\gamma = \gamma'$ in (5.16), we obtain the bounds (5.16) with larger γ'. By using this argument recursively, we can improve the order of the estimate $T_t\mathbf{1}(x) \le C_{V,t,\gamma}(1+|x|)^{-\gamma'}$. If $\gamma' = d$ occurs after some step, then we take $\gamma = d - \frac{\alpha}{2}$ in the next one. On iteration, after $\lfloor 2 + \frac{d}{\alpha} \rfloor$ steps $T_t\mathbf{1}(x) \le C_{V,t}(1+|x|)^{-d-\alpha}$ follows, for all $x \in \mathbf{R}^d$.

(Step 4) To complete the proof of the theorem we prove the implication (4) ⇒ (1). By the bound

$$u(t,x,y) = \int_{\mathbf{R}^d}\int_{\mathbf{R}^d} u\left(\frac{t}{3},x,z\right)u\left(\frac{t}{3},z,v\right)u\left(\frac{t}{3},v,y\right)dvdz \le C_{V,t}T_{t/3}\mathbf{1}(x)T_{t/3}\mathbf{1}(y),$$

it suffices to show that $T_t\mathbf{1}(x) \le C_{V,t}\varphi_1(x)$, for $x \in \mathbf{R}^d$ and $t > 0$. Put $D = B(x,1)$ and $r = \frac{|x|}{2}$. Let $R > 3$ be sufficiently large so that $D \cap \operatorname{supp}(V_-) = \emptyset$ for $|x| > R$. If $\lambda_1 > 0$, we choose $\eta = 0$, if $\lambda_1 = 0$, we choose $\eta = 1$, and if $\lambda_1 < 0$ we choose $\eta = -2\lambda_1$. In all these cases $\eta + \lambda_1 > 0$. Then we have

$$T_t\mathbf{1}(x) = e^{\eta t}e^{-\eta t}T_t\mathbf{1}(x) = C_{V,t}\left(\mathbf{E}^x\left[\frac{t}{2} < \tau_D; e_{V_\eta}(t)\right] + \mathbf{E}^x\left[\frac{t}{2} \ge \tau_D; e_{V_\eta}(t)\right]\right), \tag{5.17}$$

where $V_\eta = V + \eta$. We start by estimating the first expectation in (5.17). Note that

$$v_{D,\eta}(x) = \mathbf{E}^x \left[\int_0^{\tau_D} e^{-\int_0^v V_\eta(X_s)ds} dv \right] \geq \mathbf{E}^x \left[\frac{t}{4} < \tau_D; \int_0^{\frac{t}{4}} e^{-\int_0^v V_\eta(X_s)ds} dv \right]$$
$$\geq \mathbf{E}^x \left[\frac{t}{4} < \tau_D; \frac{t}{4} e^{-\int_0^{\frac{t}{4}} V_\eta(X_s)ds} \right] = \frac{t}{4} \mathbf{E}^x \left[\frac{t}{4} < \tau_D; e_{V_\eta}\left(\frac{t}{4}\right) \right].$$

Using this, Lemma 5.3 (2) and condition (4) of Theorem 5.1, we obtain
(5.18)

$$\mathbf{E}^x \left[\frac{t}{2} < \tau_D; e_{V_\eta}(t) \right] \leq C_{V,t} \mathbf{E}^x \left[\frac{t}{4} < \tau_D; e_{V_\eta}\left(\frac{t}{4}\right) \right] \sup_{y \in D} T_{3t/4}\mathbf{1}(y) \leq C_{V,t} v_{D,\eta}(x)(1+|x|)^{-d-\alpha}.$$

For the second expectation in (5.17) a combination of Lemma 5.3 (1), (3.10), (3.12), condition (4) and (3.11) yields

$$\mathbf{E}^x \left[\frac{t}{2} \geq \tau_D; e_{V_\eta}(t) \right] \leq C_{V,t} \mathbf{E}^x \left[e_{V_\eta}(\tau_D) \mathbf{E}^{X_{\tau_D}} \left[e_{V_\eta}\left(\frac{t}{2}\right) \right] \right]$$
$$= C_{V,t} \mathbf{E}^x \left[X_{\tau_D} \in B(x,r); e_{V_\eta}(\tau_D) \mathbf{E}^{X_{\tau_D}} \left[e_{V_\eta}\left(\frac{t}{2}\right) \right] \right]$$
$$+ C_{V,t} \mathbf{E}^x \left[X_{\tau_D} \in B(x,r)^c; e_{V_\eta}(\tau_D) \mathbf{E}^{X_{\tau_D}} \left[e_{V_\eta}\left(\frac{t}{2}\right) \right] \right]$$
$$\leq C_{V,t} \left(u_{D,\eta}(x) \sup_{y \in B(x,r)} T_{t/2}\mathbf{1}(y) + \int_D G_D^{V_\eta}(x,y) \int_{B(x,r)^c} T_{t/2}\mathbf{1}(z)|z-y|^{-d-\alpha} dz dy \right)$$
$$\leq C_{V,t} \left(v_{D,\eta}(x)(1+|x|)^{-d-\alpha} + v_{D,\eta}(x) \int_{B(x,r)^c} (1+|z|)^{-d-\alpha}|z-x|^{-d-\alpha} dz \right)$$
$$\leq C_{V,t} v_{D,\eta}(x)(1+|x|)^{-d-\alpha}.$$

By (5.17) and (5.18) this gives $T_t\mathbf{1}(x) \leq C_{V,t} v_{D,\eta}(x)(1+|x|)^{-d-\alpha}$ for $|x| > R$. Thus by Theorem 4.1 we obtain $T_t\mathbf{1}(x) \leq C_{V,t}\varphi_0(x)$ for $|x| > R$. Since φ_0 is continuous and strictly positive, we have that $\inf_{z \in B(0,R)} \varphi_0(z) > 0$. Hence for $|x| \leq R$ we have

$$T_t\mathbf{1}(x) \leq C_{V,t} \inf_{z \in B(0,R)} \varphi_1(z) \leq C_{V,t}\varphi_0(x),$$

which completes the proof of the theorem. □

Using Theorem 5.1 a sufficient condition for IUC in terms of the behaviour of the potential V at infinity is as follows.

Theorem 5.2 (**Sufficient condition for IUC**). *Let V be a Kato-decomposable potential. If $\lim_{|x|\to\infty} \frac{V(x)}{\log |x|} = \infty$, then the operators T_t are compact and the semigroup $\{T_t : t \geq 0\}$ is intrinsically ultracontractive.*

Proof. By the assumption we have also $\lim_{|x|\to\infty} V(x) = \infty$. Hence by Lemma 4.2 each T_t is compact. There exists $R_1 > 0$ large enough such that $\operatorname{supp}(V_-) \subset B(0, R_1)$. Thus

$$\mathbf{E}^x[t < \tau_{\overline{B}(0,r)^c}; e_V(t)] \leq e^{-\lambda(r)t},$$

for every $x \in \overline{B}(0,r)^c$, $r > R_1$, where $\lambda(r) = \inf_{y \in \overline{B}(0,r)^c} V(y)$. Moreover, by assumption, there is $R \geq R_1$ such that $\lambda(r) \geq \frac{d+\alpha}{t} \log(1+r)$, for $r > R$. Thus

$$e^{-\lambda(r)t} \leq C(1+r)^{-d-\alpha}$$

for $r > R$. Whenever $r \leq R$ and $x \in \overline{B}(0,r)^c$, then

$$\mathbf{E}^x[t < \tau_{\overline{B}(0,r)^c}; e_V(t)] \leq C_{V,t} = C_{V,t}(1+R)^{d+\alpha}(1+R)^{-d-\alpha} \leq C_{V,t}(1+r)^{-d-\alpha}.$$

Hence

$$\mathbf{E}^x[t < \tau_{\overline{B}(0,r)^c}; e_V(t)] \leq C_{V,t}(1+r)^{-d-\alpha}$$

for $x \in \overline{B}(0,r)^c$, $r > 0$. The assertion follows now by Theorem 5.1. □

Theorem 5.3 (**Necessary condition for IUC**). *Let V be a Kato-decomposable potential such that $V(x) \to \infty$ as $|x| \to \infty$. If the semigroup $\{T_t : t \geq 0\}$ is intrinsically ultracontractive, then for any $\epsilon \in (0,1]$*

$$\lim_{|x|\to\infty} \frac{\sup_{y \in B(x,\epsilon)} V(y)}{\log|x|} = \infty.$$

Proof. Set $r = \frac{|x|}{2}$ for $|x| \geq 2$ and $D = B(x,\epsilon)$ for an arbitrary $0 < \epsilon \leq 1$. By (3) of Theorem 5.1 we have for $|x| \geq 2$ and $t > 0$ that

$$\mathbf{P}^x(t < \tau_D)e^{-\sup_{y\in D} V(y)t} \leq \mathbf{E}^x[t < \tau_D; e_V(t)] \leq \mathbf{E}^x[t < \tau_{\overline{B}(0,r)^c}; e_V(t)] \leq C_{V,t}(1+r)^{-d-\alpha}.$$

Hence for $0 < t \leq 1$ and $|x| \geq 2$,

$$\mathbf{P}^0(1 < \tau_{B(0,\epsilon)})e^{-\sup_{y\in D} V(y)t} \leq C_{V,t}|x|^{-d-\alpha}.$$

It follows that

$$e^{-\sup_{y\in D} V(y)t} \leq C_{V,t,\epsilon}|x|^{-d-\alpha}$$

and thus

$$\frac{\sup_{y\in D} V(y)}{\log|x|} \geq \frac{\alpha+d}{t} - \frac{C_{V,t,\epsilon}}{t\log|x|}.$$

This implies $\liminf_{|x|\to\infty} \frac{\sup_{y\in D} V(y)}{\log|x|} \geq \frac{\alpha+d}{t}$, for any $0 < t \leq 1$. □

For potentials V comparable on unit balls outside a compact set we obtain the following result. This brings us closest so far to the sharp result in Remark 5.2 for fractional Schrödinger semigroups.

Corollary 5.2 (**Borderline case**). *Let V be a Kato decomposable potential such that $V(x) \to \infty$ as $|x| \to \infty$. Suppose there exist $R > 0$ such that $B(0,R)^c \cap \operatorname{supp}(V_-) = \emptyset$, and a constant $M_V > 0$ such that for every $|x| \geq R$ and $y \in B(x,1)$*

$$V(y) \leq M_V\, V(x) \tag{5.19}$$

holds. Then the semigroup $\{T_t : t \geq 0\}$ is IUC if and only if $\lim_{|x|\to\infty} \frac{V(x)}{\log|x|} = \infty$.

Proof. Direct consequence of Theorems 5.2 and 5.3, and (5.19). □

5.4. **Asymptotic intrinsic ultracontractivity of fractional Schrödinger semigroups**

We will show in this section that in case of symmetric stable processes AIUC is a weaker property than IUC. In particular, we prove that in contrast to classic case the borderline logarithmic growth of the potential implies AIUC. However, we do not know whether in the case of Brownian motion AIUC is a weaker property than IUC or not.

Theorem 5.4 (**Characterization of AIUC**). *Let V be a Kato-decomposable potential such that $V(x) \to \infty$ as $|x| \to \infty$. The following conditions are equivalent:*

(1) *The semigroup* $\{T_t : t \geq 0\}$ *is asymptotically intrinsically ultracontractive.*
(2) *There exists* $t_0 > 0$ *such that for any* $t > t_0$ *there is a constant* $C_{V,t}$ *such that for all* $x, y \in \mathbf{R}^d$

$$u(t, x, y) \leq C_{V,t}(1 + |x|)^{-d-\alpha}(1 + |y|)^{-d-\alpha}.$$

(3) *There exists* $t_0 > 0$ *such that for any* $t > t_0$ *there is a constant* $C_{V,t}$ *such that for all* $r > 0$, $x \in \overline{B}(0, r)^c$

$$\mathbf{E}^x[t < \tau_{\overline{B}(0,r)^c}; e_V(t)] \leq C_{V,t}(1 + r)^{-d-\alpha}.$$

(4) *There exists* $t_0 > 0$ *such that for any* $t > t_0$ *there is a constant* $C_{V,t}$ *such that for all* $x \in \mathbf{R}^d$

$$T_t\mathbf{1}(x) \leq C_{V,t}(1 + |x|)^{-d-\alpha}.$$

Proof. The proof runs the same way as the proof of Theorem 5.1. □

Theorem 5.5 (**Sufficient condition for AIUC**). *Let* V *be a Kato-decomposable potential such that* $\lim_{|x|\to\infty} V(x) = \infty$. *If there exists* $R > 1$ *and* $C_{V,R} > 0$ *such that for all* $|x| > R$

$$\frac{V(x)}{\log |x|} \geq C_{V,R}, \tag{5.20}$$

then each operator T_t, $t > 0$, *is compact and the semigroup* $\{T_t : t \geq 0\}$ *is asymptotically intrinsically ultracontractive.*

Proof. By condition (5.20) we have $\lim_{|x|\to\infty} V(x) = \infty$ and by Lemma 4.2 each T_t is compact. Let $r > R$. Denote $g(r) = \inf_{x\in B(0,r)^c} V(x)$ and fix $t_0 = \frac{\alpha+d}{C_{V,R}}$. By assumption, for all $t \geq t_0$ we have

$$g(r) \geq C_{V,R} \log(r) \geq \frac{d + \alpha}{t} \log r,$$

which gives

$$e^{-g(r)t} \leq C(1 + r)^{-d-\alpha}$$

for $r > R$. Since

$$\mathbf{E}^x[t < \tau_{\overline{B}(0,r)^c}; e_V(t)] \leq e^{-g(r)t},$$

for every $x \in \overline{B}(0, r)^c$, $r > R$, we obtain

$$\mathbf{E}^x[t < \tau_{\overline{B}(0,r)^c}; e_V(t)] \leq C(1 + r)^{-d-\alpha},$$

for every $x \in \overline{B}(0, r)^c$, $r > R$, and $t \geq t_0 = \frac{\alpha+d}{C_{V,R}}$.

If $r \leq R$ and $x \in \overline{B}(0, r)^c$, then

$$\mathbf{E}^x[t < \tau_{\overline{B}(0,r)^c}; e_V(t)] \leq C_{V,t} = C_{V,t}(1 + R)^{d+\alpha}(1 + R)^{-d-\alpha} \leq C_{V,t}(1 + r)^{-d-\alpha}.$$

Hence there exists $t_0 > 0$ such that for $t \geq t_0$ and $x \in \overline{B}(0, r)^c$, $r > 0$, we have

$$\mathbf{E}^x[t < \tau_{\overline{B}(0,r)^c}; e_V(t)] \leq C_{V,t}(1 + r)^{-d-\alpha}.$$

The assertion follows now by Theorem 5.4. □

Theorem 5.6 (**Necessary condition for AIUC**). *Let* V *be a Kato-decomposable potential such that* $V(x) \to \infty$ *as* $|x| \to \infty$. *If the semigroup* $\{T_t : t \geq 0\}$ *is asymptotically intrinsically ultracontractive, then there exists a constant* $C_V > 0$ *such that for every* $\varepsilon \in (0, 1]$ *there is* $R_\varepsilon > 2$ *such that for all* $|x| > R_\varepsilon$

$$\frac{\sup_{y\in B(x,\varepsilon)} V(y)}{\log |x|} \geq C_V. \tag{5.21}$$

Proof. Set $r = \frac{|x|}{2}$ for $|x| \geq 2$ and $D = B(x, \epsilon)$ for an arbitrary $0 < \varepsilon \leq 1$. By (3) of Theorem 5.4 there exists $t_0 > 0$ such that for $|x| \geq 2$ we have

$$\begin{aligned} \mathbf{P}^x(t_0 < \tau_D) e^{-\sup_{y \in D} V(y) t_0} &\leq \mathbf{E}^x[t_0 < \tau_D; e_V(t_0)] \\ &\leq \mathbf{E}^x[t_0 < \tau_{\overline{B}(0,r)^c}; e_V(t_0)] \leq C_{V,t_0}(1+r)^{-d-\alpha}. \end{aligned}$$

Hence for $|x| \geq 2$,

$$\mathbf{P}^0(t_0 < \tau_{B(0,\epsilon)}) e^{-\sup_{y \in D} V(y) t_0} \leq C_{V,t_0} |x|^{-d-\alpha}.$$

It follows that

$$e^{-\sup_{y \in D} V(y) t_0} \leq \frac{C_{V,t_0}}{\mathbf{P}^0(t_0 < \tau_{B(0,\epsilon)})} |x|^{-d-\alpha}$$

and thus

$$\frac{\sup_{y \in D} V(y)}{\log |x|} \geq \frac{1}{t_0} \left(\alpha + d - \frac{\log \left(\frac{C_{V,t_0}}{\mathbf{P}^0(t_0 < \tau_{B(0,\epsilon)})} \right)}{\log |x|} \right).$$

Now it is enough to choose $R_\varepsilon > 2$ such that for $|x| > R_\varepsilon$ we have

$$\frac{\alpha + d}{2} \geq \frac{\log \left(\frac{C_{V,t_0}}{\mathbf{P}^0(t_0 < \tau_{B(0,\epsilon)})} \right)}{\log |x|}.$$

□

Corollary 5.3. *Let V be a Kato-decomposable potential such that $V(x) \to \infty$ as $|x| \to \infty$. Suppose there exist $R > 0$ such that $B(0,R)^c \cap \operatorname{supp}(V_-) = \emptyset$, and a constant $M_V > 0$ such that for every $|x| \geq R$ and $y \in B(x,1)$ inequality (5.19) holds. Then the semigroup $\{T_t : t \geq 0\}$ is AIUC if and only if there exists $R > 0$ and $C_{V,R} > 0$ such that*

$$\frac{V(x)}{\log |x|} \geq C_{V,R}.$$

Proof. Straightforward consequence of Theorems 5.5 and 5.6, and (5.19). □

From the above it follows that IUC is a stronger property than AIUC. Indeed, we have

Example 5.2. Let

$$V(x) = \log |x| \mathbf{1}_{\{|x|>1\}}(x) - \frac{1}{|x|^{\alpha/2}} \mathbf{1}_{\{|x| \leq 1\}}(x).$$

Then the Feynman-Kac semigroup $\{T_t : t \geq 0\}$ corresponding to $(-\Delta)^{\alpha/2} + V$ is AIUC but it is not IUC.

6. Gibbs measures for symmetric α-stable processes

6.1. Construction and existence

In this section we prove that provided Assumption 4.1 holds, there exists a probability measure μ on $(D_{\mathrm{r}}(\mathbf{R}, \mathbf{R}^d), \mathcal{B}(D_{\mathrm{r}}(\mathbf{R}, \mathbf{R}^d))$ such that the left hand side of (3.1) in Theorem 3.3 can be represented in terms of an expectation with respect to it, i.e., for $f, g \in L^2(\mathbf{R}^d)$ and Kato-decomposable potential V

$$(f, e^{-t\big((-\Delta)^{\alpha/2}+V\big)}g) = \mathbf{E}_\mu\left[\overline{f(X_0)}g(X_t)\right], \quad t \geq 0. \tag{6.1}$$

We will identify the probability measure μ as a Gibbs measure with respect to the stable bridge measure and potential V, and analyze its uniqueness and support properties. The properties of μ will be important information when in forthcoming work we will construct Gibbs measures for perturbations of the fractional Schrödinger operator by other operators, which can be used to describe ground state properties of so obtained operators.

Let $I \subset \mathbf{R}$ be an interval, and denote by $\Omega_I = D_{\rm r}(I, \mathbf{R}^d)$ the space of right continuous functions from I to $\mathbf{R}^d$ with left limits. Consider a two-sided α-stable process $(X_t)_{t\in\mathbf{R}}$ with path space Ω as defined in Section 2.2. We denote by $\mathcal{F}_I$ the σ-field generated by the coordinate process $\omega(t)$, $\omega \in \Omega_I$, $t \in I$, and use the notations $\Omega := \Omega_{\mathbf{R}}$, $\mathcal{F} := \mathcal{F}_{\mathbf{R}}$, $\mathcal{F}_T := \mathcal{F}_{[-T,T)}$, $\mathcal{T}_T := \mathcal{F}_{[-T,T)^c}$.

Without restricting generality we consider $I = [-T, T)$, $T > 0$. Let $\bar\omega \in \Omega$, and consider Dirac point measure $\delta_T^{\bar\omega}$ on $\Omega_{[-T,T)^c}$ concentrated on $\bar\omega \in \Omega$. For every $T > 0$ we define a measure on $(\Omega, \mathcal{F})$ by

$$\nu_T^{\bar\omega} := \nu_T^{\bar\omega(-T),\bar\omega(T)} \otimes \delta_T^{\bar\omega}\,. \tag{6.2}$$

In what follows we consider the family of measures $(\nu_T^{\bar\omega})_{T>0}$ as reference measure.

Let V be a Kato-decomposable potential and define

$$Z_T(x,y) := \int_\Omega e^{-\int_{-T}^T V(X_s(\omega))ds} d\nu_T^{x,y}(\omega)\,, \tag{6.3}$$

for all $T > 0$ and all $x, y \in \mathbf{R}^d$. By Lemma 3.3 (5) we have

$$Z_T(x,y) = u(2T, x, y) < \infty, \quad x, y \in \mathbf{R}^d,\ T > 0.$$

For every $T > 0$ we define the conditional probability kernel

$$\mu_T(A, \bar\omega) = \frac{1}{Z_T(\bar\omega(-T), \bar\omega(T))}\int_\Omega \mathbf{1}_A(\omega)e^{-\int_{-T}^T V(X_s(\omega))ds} d\nu_T^{\bar\omega}(\omega), \quad A \in \mathcal{F},\ \bar\omega \in \Omega. \tag{6.4}$$

We refer to $\bar\omega$ as a boundary path configuration.

Definition 6.1 (**Gibbs measure**). *A probability measure μ on $(\Omega, \mathcal{F})$ is called (an infinite volume)* Gibbs measure *for the symmetric α-stable process $(X_t)_{t\in\mathbf{R}}$ under potential V if for every $A \in \mathcal{F}$ and every $T > 0$ the function $\bar\omega \mapsto \mu_T(A, \bar\omega)$ is a version of the conditional probability $\mu(A|\mathcal{T}_T)$, i.e.,*

$$\mu(A|\mathcal{T}_T)(\bar\omega) = \mu_T(A, \bar\omega), \quad A \in \mathcal{F},\ T > 0,\ a.e.\ \bar\omega \in \Omega. \tag{6.5}$$

A probability measure μ with property (6.5) is said to satisfy the *Dobrushin-Lanford-Ruelle (DLR) equations.*

Definition 6.2 (**Finite volume Gibbs measure**). *A probability measure μ_T on $(\Omega, \mathcal{F}_T)$ is called a* finite volume Gibbs measure *for the symmetric α-stable process $(X_t)_{t\in\mathbf{R}}$ under potential V and volume $[-T, T]$ if for every $A \in \mathcal{F}_T$ and every $0 < S < T$ the function $\bar\omega \mapsto \mu_S(A, \bar\omega)$ is a version of the conditional probability $\mu_T(A|\mathcal{T}_S)$.*

Using (3.1), (3.7), (5.4) and (5.6) we obtain

(6.6)
$$
\begin{aligned}
&(f, e^{-(t-s)\left((-\Delta)^{\alpha/2}+V\right)}g) \\
&= e^{\lambda_0(t-s)} \int_{\mathbf{R}^d} dx \varphi_0(x) \int_{\mathbf{R}^d} dy \varphi_0(y) \int_\Omega e^{-\int_s^t V(X_r(\omega))dr} \overline{f(X_0(\omega))} g(X_t(\omega)) d\nu_{s,t}^{x,y}(\omega)
\end{aligned}
$$

Making use of the expression in the right hand side of (6.6) we define a family of probability measures $(\mu_T)_{T>0}$ by

$$
\mu_T(A) := e^{2\lambda_0 T} \int_{\mathbf{R}^d} dx\, \varphi_0(x) \int_{\mathbf{R}^d} dy\, \varphi_0(y) \int_\Omega \mathbf{1}_B(\omega) e^{-\int_{-T}^T V(X_t(\omega))dt} d\nu_T^{x,y}(\omega), \quad A \in \mathcal{F}_T. \tag{6.7}
$$

Notice that for every $T > 0$ the measure μ_T is indeed a probability measure on Ω. By (3.7) we have

$$
\mu_T(\Omega) = e^{2\lambda_0 T} \int_{\mathbf{R}^d} \varphi_0(x) \left(\int_{\mathbf{R}^d} \varphi_0(y) u(2T, x, y) dy \right) dx = \int_{\mathbf{R}^d} \varphi_0^2(x) dx = \|\varphi_0\|_2^2 = 1.
$$

Lemma 6.1. *For every $T > 0$ the measure μ_T given by* (6.7) *is a finite volume Gibbs measure corresponding to the symmetric α-stable process under the potential V.*

Proof. We show that for every $0 < S < T$, $\bar\omega \in \Omega$ and $C \in \mathcal{F}_T$, the function $\bar\omega \mapsto \mu_S(C, \bar\omega)$ is a version of the conditional probability $\mu_T(C|\mathcal{T}_S)$.

Let $0 < S < T$, $A \in \mathcal{F}_S$, $B_1 \in \mathcal{F}_{[-T,-S)}$, $B_2 \in \mathcal{F}_{[S,T)}$, $B = B_1 \cap B_2 \in \mathcal{F}_{[-T,T)\setminus[-S,S)}$. By a monotone class argument, it suffices to consider sets of the form $A \cap B$. In order to show $\mu_T(\mu_S(A \cap B, \cdot)) = \mu_T(A \cap B)$ first note that since $\nu_T^{\xi,\eta}(\{\bar\omega(-T) \neq \xi\}) = \nu_T^{\xi,\eta}(\{\bar\omega(T) \neq \eta\}) = 0$, we have

$$
\int_\Omega e^{-\int_{-S}^S V(X_s(\bar\omega))\, ds} \mu_S(A, \bar\omega)\, d\nu_S^{\xi,\eta}(\bar\omega) = \int_\Omega e^{-\int_{-S}^S V(X_s(\bar\omega))\, ds} \mathbf{1}_A(\bar\omega)\, d\nu_S^{\xi,\eta}(\bar\omega).
$$

Then the Markov property of the process yields

$$
\begin{aligned}
&\int_\Omega e^{-\int_{-T}^T V(X_s(\bar\omega))\, ds} \mu_S(A \cap B, \bar\omega)\, d\nu_T^{x,y}(\bar\omega) \\
&= \int_{\mathbf{R}^d \times \mathbf{R}^d} \left(\int_\Omega e^{-\int_{-T}^{-S} V(X_s(\bar\omega))\, ds} \mathbf{1}_{B_1}(\bar\omega)\, d\nu_{[-T,-S)}^{x,\xi}(\bar\omega) \right) \left(\int_\Omega e^{-\int_{-S}^S V(X_s(\bar\omega))\, ds} \mu_S(A, \bar\omega)\, d\nu_S^{\xi,\eta}(\bar\omega) \right) \\
&\quad \times \left(\int_\Omega e^{-\int_S^T V(X_s(\bar\omega))\, ds} \mathbf{1}_{B_2}(\bar\omega)\, d\nu_{[S,T)}^{\eta,y}(\bar\omega) \right) d\xi d\eta \\
&= \int_\Omega e^{-\int_{-T}^T V(X_s(\bar\omega))\, ds} \mathbf{1}_{A\cap B}(\bar\omega)\, d\nu_T^{x,y}(\bar\omega)
\end{aligned}
$$

for all $x, y \in \mathbf{R}^d$. By integrating in the above equality with respect to the measure $\varphi_0(x)\varphi_0(y)dxdy$ over $\mathbf{R}^d \times \mathbf{R}^d$, we plainly obtain

$$
\int_\Omega \mu_S(A \cap B, \bar\omega)\, d\mu_T(\bar\omega) = \mu_T(A \cap B). \tag{6.8}
$$

As $\bar\omega \mapsto \mu_S(C, \bar\omega)$ is $\mathcal{T}_S$-measurable, the lemma is proven. □

Lemma 6.2. *The measures $(\mu_T)_{T>0}$ form a consistent family of probability measures.*

Proof. Let $0 < T_1 < T_2$ and $B \in \mathcal{F}_{T_1}$. We show that

$$
\mu_{T_1}(B) = \mu_{T_2}(B).
$$

By the Markov property of the symmetric α-stable process and (3.7) we obtain

$$\begin{aligned}
&\int_{\mathbf{R}^d} dx\, \varphi_0(x) \int_{\mathbf{R}^d} dy\, \varphi_0(y) \int_\Omega \mathbf{1}_B(\omega) e^{-\int_{-T_2}^{T_2} V(X_s(\omega))ds} d\nu_{T_2}^{x,y}(\omega) \\
&\quad = \int_{\mathbf{R}^d} dx\, \varphi_0(x) \int_{\mathbf{R}^d} dy\, \varphi_0(y) \int_{\mathbf{R}^d} dz \int_{\mathbf{R}^d} dv \int_\Omega e^{-\int_{-T_2}^{-T_1} V(X_s(\omega))ds} d\nu_{[-T_2,-T_1)}^{x,z}(\omega) \\
&\quad\quad \times \int_\Omega \mathbf{1}_B(\omega) e^{-\int_{-T_1}^{T_1} V(\omega(s))ds} d\nu_{[-T_1,T_1)}^{z,v}(\omega) \int_\Omega e^{-\int_{T_1}^{T_2} V(X_s(\omega))ds} d\nu_{[T_1,T_2)}^{v,y}(\omega) \\
&\quad = \int_{\mathbf{R}^d} dz \int_{\mathbf{R}^d} dv \int_\Omega \mathbf{1}_B(\omega) e^{-\int_{-T_1}^{T_1} V(X_s(\omega))ds} d\nu_{[-T_1,T_1)}^{z,v}(\omega) \\
&\quad\quad \times \int_{\mathbf{R}^d} dx\, \varphi_0(x) u(T_2 - T_1, x, z) \int_{\mathbf{R}^d} dy\, \varphi_0(y) u(T_2 - T_1, v, y) \\
&\quad = \int_{\mathbf{R}^d} dz\, \varphi_0(z) \int_{\mathbf{R}^d} dv\, \varphi_0(v) \int_\Omega \mathbf{1}_B(\omega) e^{-\int_{-T_1}^{T_1} V(X_s(\omega))ds} d\nu_{T_1}^{z,v}(\omega).
\end{aligned}$$

□

Define the set function

$$\mu(B) = \mu_T(B), \quad B \in \mathcal{F}_T,\ T > 0. \tag{6.9}$$

It is easy to check that μ is a finitely additive set function on the σ-field $\bigcup_{T>0} \mathcal{F}_T$ such that $\mu(\Omega) = 1$. This is an immediate consequence of the definition of μ and Lemma 6.2.

Lemma 6.3. *The finitely additive set function μ given by (6.9) has a unique extension to a probability measure on $(\Omega, \mathcal{F})$. Moreover, μ is the stationary measure of the time reversible Markov process $(X_t)_{t\in\mathbf{R}}$.*

Proof. Since $\mathcal{F} = \sigma(\bigcup_{T>0} \mathcal{F}_T)$ it suffices to check that μ satisfies the following continuity condition. We will show that if $(B_n)_{n\in\mathbb{N}}$ is an increasing family of sets of paths from $\bigcup_{T>0} \mathcal{F}_T$ and $\bigcup_{n=1}^\infty B_n = \lim_{n\to\infty} B_n = B \in \bigcup_{T>0} \mathcal{F}_T$, then $\lim_{n\to\infty} \mu(B_n) = \mu(B)$.

Notice that for each $n \in \mathbf{N}$ there exists $T_n > 0$ such that $T_n \geq T_{n+1}$ and $B \in \mathcal{F}_{T_n}$. Moreover, there is $T > 0$ such that $B \in \mathcal{F}_T$ and $0 < T \leq T_n$, $n \in \mathbf{N}$. Thus by the definition of μ we have

$$\mu(B_n) = \mu_{T_1}(B_n) = \int_{\mathbf{R}^d} dx\, \varphi_0(x) \int_{\mathbf{R}^d} dy\, \varphi_0(y) \int_\Omega \mathbf{1}_{B_n}(\omega) e^{-\int_{-T_1}^{T_1} V(X_s(\omega))ds} d\nu_{T_1}^{x,y}(\omega).$$

Bounded convergence gives that the last integral converges to

$$\int_{\mathbf{R}^d} dx\, \varphi_0(x) \int_{\mathbf{R}^d} dy\, \varphi_0(y) \int_\Omega \mathbf{1}_B(\omega) e^{-\int_{-T_1}^{T_1} V(X_s(\omega))ds} d\nu_{T_1}^{x,y}(\omega) = \mu_{T_1}(B) = \mu(B).$$

Let $0 \leq t_1 < t_2 < t_3 < ... < t_n$ and $A_1, A_2, ..., A_n \in \mathbf{R}^d$ be Borel sets. Then it is straightforward to check that

$$\mu(\omega(t_1) \in A_1, \omega(t_2) \in A_2, ..., \omega(t_n) \in A_n) = \mu(\omega(-t_1) \in A_1, \omega(-t_2) \in A_2, ..., \omega(-t_n) \in A_n).$$

□

In the following we denote the above extension by the same μ.

Lemma 6.4. *Let μ be the stationary measure of the time reversible Markov process $(X_t)_{t\in\mathbf{R}}$ corresponding to the potential V. For every $T > 0$, $\bar{\omega} \in \Omega$ and $A \in \mathcal{F}$, $\bar{\omega} \mapsto \mu_T(A, \bar{\omega})$ is a version of the conditional probability $\mu(A|\mathcal{F}_T)(\bar{\omega})$, hence μ is a Gibbs measure for V.*

Proof. Similar arguments as in the proof of Lemma 6.1 and (6.9) imply the statement. □

6.2. **Uniqueness and support properties**

It is seen above that the measure of the time reversible Markov process $(X_t)_{t\in\mathbf{R}}$ is a Gibbs measure for the given potential V. In fact, the existence of a Gibbs measure μ follows from the existence of the ground state eigenfunction φ_0 of the operator $(-\Delta)^{\alpha/2}+V$. However, it is not clear whether there are any other probability measures on $(\Omega,\mathcal{F})$ satisfying the DLR equations for the potential V. This problem will be discussed in this section.

In the case of the Schrödinger operator $(-1/2)\Delta+V$ and Brownian motion $(B_t)_{t\in\mathbf{R}}$, the one-dimensional Ornstein-Uhlenbeck process obtained for $V(x)=\frac{1}{2}(x^2-1)$ shows that uniqueness need not hold in general (see [6, Ex. 3.1]). In fact, in this case there are uncountably many Gibbs measures supported on $C(\mathbf{R},\mathbf{R}^d)$ for this potential.

We start with two lemmas concerning the uniqueness of Gibbs measures, which were proved in [6] in the case of Gibbs measures on Brownian motion. The first lemma gives a simple criterion allowing to check if a Gibbs measure is the only one supported on a given set. We say that a probability measure P is supported on a set B if $P(B)=1$.

Lemma 6.5. *Let $\Omega^*\subset\Omega$ be measurable and ν be a Gibbs measure for the potential V such that $\nu(\Omega^*)=1$. Suppose that for every $T>0$, $B\in\mathcal{F}_T$ and $\bar\omega\in\Omega^*$, $\nu_N(B,\bar\omega)\to\nu(B)$ as $N\to\infty$, where $\nu_N(B,\bar\omega)$ is the probability kernel defined in* (6.4)*. Then ν is the only Gibbs measure for V supported on Ω^*.*

Proof. Let $\tilde\nu$ be a Gibbs measure supported on Ω^*. It is easy to check that for every $T>0$ and $B\in\mathcal{F}_T$, $\bar\omega\mapsto\tilde\nu(B|\mathcal{T}_N)(\bar\omega)$ is a backward martingale in N and thus convergent $\tilde\nu$-almost surely to $\tilde\nu(B|\mathcal{T})(\bar\omega)$, where $\mathcal{T}=\bigcap_{N\in\mathbf{N}}\mathcal{T}_N$ is the tail field. By Definition 6.1,

$$\tilde\nu(B|\mathcal{T}_N)(\bar\omega)=\nu_N(B,\bar\omega),\qquad \tilde\nu-a.s.$$

Thus for $\tilde\nu$-almost every $\bar\omega\in\Omega^*$, we have

$$\tilde\nu(B|\mathcal{T})(\bar\omega)=\lim_{N\to\infty}\tilde\nu(B|\mathcal{T}_N)(\bar\omega)=\lim_{N\to\infty}\nu_N(B,\bar\omega)=\nu(B).$$

By taking $\tilde\nu$-expectation on both sides of the above equality it follows that $\tilde\nu(B)=\nu(B)$ for every $T>0$ and $B\in\mathcal{F}_T$. Thus $\tilde\nu=\nu$. ☐

Note that the condition

$$\lim_{N\to\infty}\sup_{(x,y)\in\mathbf{R}^d\times\mathbf{R}^d}\left(\left|\frac{\widetilde{u}(N-T,\bar\omega(-N),x)\widetilde{u}(N-T,y,\bar\omega(N))}{\widetilde{u}(2N,\bar\omega(-N),\bar\omega(N))}-1\right|\varphi_0(x)\varphi_0(y)\right)=0 \tag{6.10}$$

is equivalent to (6.11), which will be useful below.

The next lemma characterizes a set of path functions $\bar\omega\in\Omega$ for which the convergence $\mu_N(B,\hat\omega)\to\mu(B)$ holds. A sufficient condition is given in terms of the kernel $u(t,x,y)$ and the ground state φ_0.

Lemma 6.6. *Let $(-\Delta)^{\alpha/2}+V$ be a fractional Schrödinger operator with Kato-decomposable potential V and ground state eigenfunction φ_0. Suppose that for some $\bar\omega\in\Omega$*

$$\frac{u(N-T,\bar\omega(-N),x)u(N-T,y,\bar\omega(N))}{u(2N,\bar\omega(-N),\bar\omega(N))}\stackrel{N\to\infty}{\longrightarrow}\varphi_0(x)\varphi_0(y) \tag{6.11}$$

holds uniformly in $(x,y)\in\mathbf{R}^d\times\mathbf{R}^d$ for every $T>0$. Then for all $T>0$ and $B\in\mathcal{F}_T$, $\mu_N(B,\bar\omega)\to\mu(B)$ as $N\to\infty$.

Proof. By the Markov property of the process $(X_t)_{t\in\mathbf{R}}$ and (5) of Lemma 3.3 we have for $N > T$, $B \in \mathcal{F}_T$ and $\bar\omega \in \Omega$

$$
\begin{aligned}
\mu_N(B,\bar\omega) &= \frac{1}{Z_N(\bar\omega(-N),\bar\omega(N))}\int_{\mathbf{R}^d} dx \int_{\mathbf{R}^d} dy \left(\int_\Omega e^{-\int_{-N}^{-T} V(X_s(\omega))ds} d\nu^{\bar\omega(-N),x}_{[-N,-T)}(\omega)\right.\\
&\quad \times \left.\int_\Omega \mathbf{1}_B(\omega) e^{-\int_{-T}^{T} V(X_s(\omega))ds} d\nu^{x,y}_{[-T,T)}(\omega) \int_\Omega e^{-\int_{T}^{N} V(X_s(\omega))ds} d\nu^{y,\bar\omega(N)}_{[T,N)}(\omega)\right)\\
&= \int_{\mathbf{R}^d} dx \int_{\mathbf{R}^d} dy\, \frac{u(N-T,\bar\omega(-N),x)u(N-T,y,\bar\omega(N))}{u(2N,\bar\omega(-N),\bar\omega(N))}\\
&\quad \times \int_\Omega \mathbf{1}_B(\omega) e^{-\int_{-T}^{T} V(X_s(\omega))ds} d\nu^{x,y}_{[-T,T)}(\omega).
\end{aligned}
\tag{6.12}
$$

Put $\Omega_M := \{\omega \in \Omega : \max(|\omega_{-T}|, |\lim_{t\nearrow T}\omega_t|) < M\}$, $M \in \mathbf{N}$. Clearly, $\Omega_M \nearrow \Omega$ when $M \to \infty$. If $B \subset \Omega_M$ for some $M > 1$, then the last factor in the above integrals is a bounded function of x and y with compact support and the assertion of the lemma follows from (6.11).

Let now $B \in \mathcal{F}_T$ be arbitrary. Fix $\epsilon > 0$ and choose M large enough such that $\mu(\Omega_M^c) < \epsilon/4$. Since the claim is true for all $\mathcal{F}_T$-measurable subsets of Ω_M, in particular for $B_M = B \cap \Omega_M$ and Ω_M we find N_0 such that for all $N > N_0$ we have

$$
|\mu_N(B_M,\bar\omega) - \mu(B_M)| < \epsilon/4 \quad \text{and} \quad |\mu_N(\Omega_M,\bar\omega) - \mu(\Omega_M)| < \epsilon/4.
$$

This gives $\mu_N(\Omega_M^c,\bar\omega) < \epsilon/2$ for $N > N_0$, and hence

$$
\begin{aligned}
|\mu_N(B,\bar\omega) - \mu(B)| &= |\mu_N(B_M,\bar\omega) + \mu_N(B\backslash\Omega_M,\bar\omega) - \mu(B_M) - \mu(B\backslash\Omega_M)|\\
&\le |\mu_N(B_M,\bar\omega) - \mu(B_M)| + \mu(\Omega_M^c) + \mu_N(\Omega_M^c,\bar\omega) \le \epsilon,
\end{aligned}
$$

completing the proof. □

We now discuss the problem of uniqueness for potentials $V(x) \to \infty$ as $|x| \to \infty$. Our first main result is the following sufficient condition for uniqueness.

Theorem 6.1. *Let μ be the measure of the reversible Markov process $(X_t)_{t\in\mathbf{R}}$ obtained in Lemma 6.3. If the semigroup $\{T_t : t \ge 0\}$ is AIUC, then μ is the unique Gibbs measure for V supported on the whole space Ω.*

Proof. Lemma 5.1 implies that condition (6.10) is satisfied for every $\omega \in \Omega$. The assertion of the theorem follows as by Lemmas 6.6 and 6.5. □

Corollary 6.1 (Uniqueness criterion). *By using Theorem 5.6 we immediately conclude that if there exist $R > 0$ and $C_{V,R} > 0$ such that for all $|x| > R$*

$$
\frac{V(x)}{\log|x|} \ge C_{V,R}, \tag{6.13}
$$

holds, then μ is the unique Gibbs measure for V supported on Ω.

Since AIUC depends only on the behaviour of the potential at infinity (cf. Theorem 5.2) local singularities and perturbations on bounded sets have no effect on the uniqueness of the Gibbs measure for this class of V.

Theorem 6.2. *Let $(-\Delta)^{\alpha/2} + V$ be a fractional Schrödinger operator with a Kato-decomposable potential V, spectral gap $\Lambda > 0$ and ground state $\varphi_0 \in L^2(\mathbf{R}^d) \cap L^1(\mathbf{R}^d)$. Then μ is the unique Gibbs measure supported on*

$$
\Omega^* := \left\{\omega \in \Omega : \lim_{|N|\to\infty} \frac{e^{-\Lambda|N|}}{\varphi_0(\omega(N))} = 0\right\}.
$$

Proof. First we show that μ is in fact supported on Ω^*. By time reversibility of μ it suffices to show that for each $\epsilon > 0$

$$\mu\left(\limsup_{N\to\infty} \frac{e^{-\Lambda N}}{\varphi_0(\omega(N))} > \epsilon\right) = 0. \tag{6.14}$$

The fact that $\varphi_0 \in L^1(\mathbf{R}^d)$ and stationarity of μ give

$$\mu\left(\frac{e^{-\Lambda N}}{\varphi_0(\omega(N))} > \epsilon\right) = \mu\left(\frac{e^{-\Lambda N}}{\epsilon} > \varphi_0(\omega(0))\right) = \int \mathbf{1}_{\{\varphi_0 < e^{-\Lambda N}/\epsilon\}}(x)\varphi_0^2(x)dx \leq \frac{e^{-\Lambda N}}{\epsilon}\left\|\varphi_0\right\|_1.$$

Since the right hand side of the above inequality is summable with respect to N for every $\epsilon > 0$, the Borel-Cantelli Lemma gives (6.14) for every $\epsilon > 0$. This shows that μ is supported on Ω^*.

Now we show that μ is the only Gibbs measure with this property. Lemma 4.1 implies that for every $0 < t < N$, $N - t \geq 2$,

$$\sup_{x,y\in\mathbf{R}^d} |u(N-t,x,y) - \varphi_0(x)\varphi_0(y)| \leq C_{V,t}e^{-\Lambda N}.$$

Thus for each $\omega \in \Omega^*$ and any $x, y \in \mathbf{R}^d$ we clearly get

$$|\widetilde{u}(N-T,\omega(-N),x) - 1|\varphi_0(x) \leq C_{V,T}\frac{e^{-\Lambda N}}{\varphi_0(\omega(-N))} \to 0,$$

$$|\widetilde{u}(N-T,y,\omega(N)) - 1|\varphi_0(y) \leq C_{V,T}\frac{e^{-\Lambda N}}{\varphi_0(\omega(N))} \to 0,$$

$$|\widetilde{u}(2N,\omega(-N),\omega(N)) - 1| \leq C_{V,T}\frac{e^{-2\Lambda N}}{\varphi_0(\omega(-N))\varphi_0(\omega(N))} \to 0$$

as $N \to \infty$, which implies (6.10). It follows from Lemmas 6.6 and 6.5 that μ is the unique Gibbs measure supported on Ω^*.

□

The asymptotic behaviour of the ground state (determined by the asymptotic behaviour of the potential) allows to specify the actual support of the measure μ. By using Theorem 4.1, a more explicit description of the subspace Ω^* for a wide class of potentials can be given.

Theorem 6.3. *Let $(-\Delta)^{\alpha/2} + V$ be a fractional Schrödinger operator with Kato-decomposable potential V such that $V(x) \to \infty$ as $|x| \to \infty$. Assume that there exists a compact set $K \in \mathbf{R}^d$ (possibly empty) such that*

(1) $K^c \subset \{x \in \mathbf{R}^d : B(x,1) \cap \operatorname{supp}(V_-) = \emptyset\}$,
(2) *there is a constant $M_{V,K} \geq 1$ such that for each $x \in K^c$*

$$V_+(y) \leq M_{V,K}V_+(z), \quad y,z \in B(x,1).$$

Then μ is the unique Gibbs measure supported on

$$\Omega^* = \left\{\omega \in \Omega : \lim_{|N|\to\infty} \frac{V_+(\omega(N))|\omega_N|^{d+\alpha}\mathbf{1}_{K^c}(\omega(N))}{e^{\lambda_1|N|}} = 0\right\}.$$

Proof. By Theorem 4.1 we have that $\varphi_0(x)$ and $(V_+(x)|x|^{d+\alpha})^{-1}$ are comparable on K^c. Since $0 < C_1 \leq \varphi_0 \leq C_2 < \infty$ on K, the assertion follows from the previous theorem. □

While condition (1) in the above theorem is weak and technical, condition (2) on the regular growth of the potential at infinity is essential. We now illustrate the above results by some examples.

Example 6.1. Let $H_\alpha = (-\Delta)^{\alpha/2} + V$ be a fractional Schrödinger operator with potential

$$V(x) = C_0|x|^\delta + \frac{C_1}{|x - x_1|^{\beta_1}} - \frac{C_2}{|x - x_2|^{\beta_2}}$$

where $C_0 > 0$, $C_1, C_2 \geq 0$, $x_1, x_2 \in \mathbf{R}^d$ and $\delta > 0$, $\beta_1, \beta_2 \geq 0$. It is straightforward to check that if $\beta_1, \beta_2 < \alpha$, then V is Kato-decomposable. An immediate consequence of Theorem 6.1 is that μ is the only Gibbs measure corresponding to the process $(X_t)_{t\in\mathbf{R}}$ and the potential V supported on Ω. Moreover, by Theorem 6.3 we obtain that the measure μ is in fact supported by the subset of configuration space Ω consisting of all path functions ω such that

$$|\omega(N)| = o\left(\exp\left(\frac{\lambda_1 - \lambda_0}{\delta + d + \alpha}|N|\right)\right).$$

Example 6.2 (Potential well). Let $d = 1$, $\alpha \in (0, 1)$ and

$$V(x) = \begin{cases} -a, & x \in [-b, b] \\ 0, & x \in [-b, b]^c, \end{cases}$$

where $a, b > 0$. It is proved in [17, Th. V.1] that the operator $H_\alpha = (-\Delta)^{\alpha/2} + V$ has a spectral gap $\gamma_\alpha > 0$ and a ground state φ_0 corresponding to the eigenvalue $\lambda_0 < 0$. By using Theorems 6.2 and 4.2 we obtain that the Gibbs measure μ is uniquely supported on the full-measure subset of paths given by the growth condition

$$|\omega(N)| = o\left(\exp\left(\frac{\gamma_\alpha}{1 + \alpha}|N|\right)\right).$$

However, we do not know whether on the whole space Ω there exist other Gibbs measures or not.

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