The effect of bottom sediment transport on wave set-up

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Abstract

In this paper we augment the usual wave-averaged mean field equations commonly used to describe wave set-up and wave-induced mean currents in the near-shore zone, with an empirical sediment flux law depending only on the wave-induced mean current and mean total depth. This model allows the bottom to evolve slowly in time, and is used to examine how sediment transport affects wave set-up in the surf zone. We show that the mean bottom depth in the surf zone evolves according to a simple wave equation, whose solution predicts that the mean bottom depth decreases and the beach is replenished. Further, we show that if the sediment flux law also allows for a diffusive dependence on the beach slope then the simple wave equation is replaced by a nonlinear diffusion equation which allows a steady-state solution, the equilibrium beach profile.

Keywords: water waves, mean flow, sediment transport

1 Introduction

The action of shoaling waves, and wave breaking in the surf zone, in generating a wave-generated mean sea-level is well-known and has been extensively studied, see for instance the monographs of Mei (1983) and Svendsen (2006). The simplest model is obtained by averaging the oscillatory wave field over the wave phase to obtain a set of equations describing the evolution of the mean fields in the shoaling zone based on small-amplitude wave theory and then combining these with averaged mass and momentum equations in the surf zone, where empirical formulae are used for the breaking waves. These lead to a prediction of steady set-down in the shoaling zone, and a set-up in the surf zone. This agrees quite well with experiments and observations, see Bowen et al (1968) for instance. However, these models assume that the sea bottom is rigid, and ignore the possible effects of sand transport by the wave currents, and the wave-generated mean currents. Our purpose in this paper is to add an empirical model of sediment transport to the wave-averaged mean field equations and hence determine the effect of this extra term on wave set-up.

There is a vast literature on sediment transport due to waves, see the recent works by Caballeria *et al* (2002), Calvete *et al* (2001, 2002), Garnier *et al* (2006), Hancock et al (2008), Lane and Restrepo (2007), Restrepo (2001), Restrepo and Bona (1995) and the references therein. There are several methods to model the movement of bottom sediment by the combined action of waves and currents, and these can often be quite complicated, depending *inter alia* on the nature of the sediment, and whether the sediment is confined to the bottom boundary layer, or is suspended throughout a larger portion of the water column. Various models have been used to describe the formation of sand bars, ripples and sand waves, see for instance the afore-mentioned articles and the review article by Blondeaux (2001). For the most part, these models have assumed that the wave field is quasi-periodic and non-breaking, and often also assume that the sediment transport is due to the mean velocity field in the bottom boundary layer. However, the effect of sediment transport on wave set-up, especially in the surf zone, does not appear to have been examined in any detail. To remedy this, we take the point of view that the wave-averaged mean field equations only need to be modified by a bottom boundary condition that allows for the evolution of the bottom as sediment is moved. This leads to a single extra equation in the wave-averaged mean field model to represent the time evolution of the bottom. We shall achieve this end by adapting a relatively simple empirical law for the bottom sediment flux, based on the sediment transport models used in similar problems in the cited references above.

In section 2 we present the usual wave-averaged mean field equations that are commonly used in the literature. Then in section 3 we introduce our bottom sediment transport model and examine the consequences for wave set-up. We conclude with a discussion in section 4.

2 Background

2.1 Wave field

In this section we recall the wave-averaged mean flow and wave action equations that are commonly used to describe the near-shore circulation (see Mei 1983 or Svendsen 2006 for instance). We suppose that the depth and the mean flow are slowly-varying compared to the waves. Then we define a wave-phase averaging operator $< f >= \bar{f}$, so that we can express all quantities as a mean field and a wave perturbation, denoted by a "tilde" overbar. For instance,

$$\zeta = \bar{\zeta} + \tilde{\zeta} \,. \tag{1}$$

where ζ is the free surface elevation above the undisturbed depth $h = h(\mathbf{x})$. Then outside the surf zone, the representation for slowly-varying, small-amplitude waves is, in standard notation,

$$\tilde{\zeta}(\mathbf{x}\,,t) \sim a \cos\theta \,. \tag{2}$$

Here $a = a(\mathbf{x}.t)$ is the wave amplitude and $\theta = \theta(\mathbf{x}, t)$ is the phase, such that the wavenumber $\mathbf{k}$, frequency Ω are given by

$$\mathbf{k} = (k\,, l) = \nabla\theta \quad \Omega = -\theta_t \,. \tag{3}$$

The local dispersion relation is

$$\Omega = \omega + \mathbf{k}.\mathbf{U}\,, \quad \omega^2 = g\kappa \tanh \kappa H \tag{4}$$

$$\text{where} \quad \kappa^2 = k^2 + l^2 \,.$$

Here $\mathbf{U}(\mathbf{x}, \mathbf{t})$ is the slowly-varying depth-averaged mean current (see below), and $H(\mathbf{x}, t) = h(\mathbf{x}) + \bar{\zeta}(\mathbf{x}, t)$. To leading order, the horizontal and vertical components of the wave velocity field are respectively

$$\tilde{\mathbf{u}} \sim \frac{\mathbf{k}}{\kappa} a\Omega \frac{\cosh \kappa(z+h)}{\sinh \kappa H} \cos\theta\,, \quad \tilde{w} \sim a\Omega \frac{\sinh \kappa(z+h)}{\sinh \kappa H} \sin\theta\,. \tag{5}$$

Importantly, note that we have ignored here any reflected wave field, which is assumed to be very weak when the bottom topography is slowly varying.

The basic equations governing the wave field is then the kinematic equation for conservation of waves

$$\mathbf{k_t} + \nabla\omega = 0\,, \tag{6}$$

which is obtained from (3) by cross-differentiation, the local dispersion relation (4), and the wave action equation for the wave amplitude

$$A_t + \nabla \cdot (\mathbf{c}_g A) = 0\,. \tag{7}$$

Here $A = E/\omega$, where $E = ga^2/2$ is the wave energy per unit mass, and $\mathbf{c}_g = \nabla_{\mathbf{k}} \cdot \omega = \mathbf{U} + c_g\mathbf{k}/\kappa, (c_g = d\omega/d\kappa)$ is the group velocity.

2.2 Mean fields

The equations governing the mean fields are obtained by averaging the depth-integrated Euler equations over the wave phase. Thus the averaged equation for conservation of mass is

$$H_t + \nabla \cdot (H\mathbf{U}) = 0\,. \tag{8}$$

Here $H = h + \bar{\zeta}$ where $h = h(\mathbf{x})$ is the time-independent undisturbed depth. For the velocity field we proceed in a slightly different way, that is we define

$$\mathbf{u} = \mathbf{U} + \mathbf{u}'\,, \tag{9}$$

where we define $\mathbf{U}$ so that the mean momentum density is given by

$$\mathbf{M} = H\mathbf{U} \; = < \int_{-h}^{\zeta} \mathbf{u}\, dz >\,, \tag{10}$$

But now we need to note that $\mathbf{u}'$ does not necessarily have zero mean, and that $\mathbf{U}$ and $\bar{\mathbf{u}}$ are not necessarily the same. Indeed, from (9) and (10) we get that

$$\bar{\mathbf{u}} = \mathbf{U} + < \mathbf{u}' >\,, \quad \text{and} \quad < \int_{-h}^{\zeta} \mathbf{u}'\, dz > = 0\,.$$

But $\mathbf{u}' = \tilde{\mathbf{u}} + O(a^2)$, so that $< \mathbf{u}' >$ is $O(a^2)$ and it follows that, correct to second order in wave amplitude,

$$\mathbf{M} = H\bar{\mathbf{u}} + \mathbf{M}_w \,, \quad \text{where} \quad \mathbf{M}_w = -H < \mathbf{u}' > = < \zeta \, \tilde{\mathbf{u}}(x, 0, t) > = \frac{E}{\omega} \mathbf{k} \,. \quad (11)$$

The term $\mathbf{M}_w$ in (11) is called the wave momentum, and can be expressed as $\mathbf{M}_w = H\mathbf{U}_s$ where $\mathbf{U}_s$ is the Stokes drift velocity. It follows that $\mathbf{U}$ is in fact the Lagrangian mean flow.

Next, averaging the depth-integrated horizontal momentum equation yields (Mei 1983)

$$(H\mathbf{U})_t + \nabla.(H\mathbf{U}\mathbf{U}) \; = \; -\nabla. < \int_{-h}^{\zeta} \mathbf{u}'\mathbf{u}' + p\mathbf{I} \, dz > + < p(z = -h) > \nabla h \,.$$

Next an estimate of the bottom pressure term is made by averaging the vertical momentum equation to get

$$< p(z = -h) > - \; g(\bar{\zeta} + h) \; = \nabla. < \int_{-h}^{\zeta} w \, \mathbf{u} \, dz > + < \int_{-h}^{\zeta} w \, dz >_t \; .$$

For slowly-varying small-amplitude waves, the integral terms on the right-hand side may be neglected, and so $< p(z = -h) > \approx \; g(\bar{\zeta} + h)$. Using this in the averaged horizontal momentum equation, and replacing the pressure p with the dynamic pressure $q = p + (z - \bar{\zeta})$ yields

$$(H\mathbf{U})_t + \nabla.(H\mathbf{U}\mathbf{U}) \; = \; -\nabla.\mathbf{S} - gH\nabla\bar{\zeta} \qquad (12)$$

$$\text{where} \quad \mathbf{S} = < \int_{-h}^{\zeta} [\mathbf{u}\mathbf{u} + q\mathbf{I}] dz > - < \frac{g}{2} \tilde{\zeta}^2 > \mathbf{I} \,. \qquad (13)$$

Here $\mathbf{S}$ is the radiation stress tensor. In the absence of any background current, so that $\mathbf{U}$ is $O(a^2)$, we may use the linearized expressions (2, 5) to find that

$$\mathbf{S} \approx \mathbf{c}_g \mathbf{k} \frac{E}{\omega} + E[\frac{c_g}{c} - \frac{1}{2}]\mathbf{I} \,. \qquad (14)$$

where the phase speed $c = \omega/\kappa$, correct to second order in the wave amplitude.

. In summary, the mean field equations to be solved are the averaged equation for conservation of mass (8) and the averaged equation for conservation of horizontal momentum (12), where the radiation stress tensor is given by (14). In the shoaling zone outside the surf zone, defined below, the wave energy E is found by the procedure described in section 2.1, that is the equation for the conservation of waves (6) combined with the dispersion relation (4), the wave action equation (7), In the surf zone, an empirical expression for the the wave energy E is used, see section 2.4 below. In order to study wave set-up, we henceforth assume that $h = h(x)$ depends only on the offshore co-ordinate $x > 0$, where the undisturbed shoreline is at $x = 0$ defined by $h(0) = 0$. Further, in the near-shore region, including all of the surf zone, we shall assume that $h_x > 0$. Then we also assume that there is no transverse (y) dependence in all mean variables, and that $\mathbf{U} = (U, 0)$.

2.3 Shoaling zone

This is defined to be the region $x > x_b$ where x_b is defined below at the end of this paragraph. The surf zone, where wave breaking occurs, is $x < x_b$ and is discussed in the next section 2.4. We seek steady solutions for which there is no dependence on t. It then follows from the mean mass equation (8) that HU is constant, in both the shoaling and the surf zone. Since $H = 0$ at the shoreline, it follows that we can set $U = 0$ everywhere. Then in the dispersion relation (4) which holds in the shoaling zone, $\Omega = \omega$. From the equation for conservation of waves (6) we see that the frequency ω and the transverse wavenumber l are constants, and the the offshore wavenumber k is then determined from the dispersion relation (4). As is well-known, it then follows that as $H \to 0$, $|k| \to \infty$, that is the waves refract towards the onshore direction, where we assume that the waves are propagating towards the shoreline so that $k < 0$. The wave action equation (7) reduces to Ec_g is constant. Near the shore, we can assume that the shallow water approximation holds and then $c_g \approx (gh)^{1/2}$, so that

$$a^2 h^{1/2} \approx F_0 \,. \tag{15}$$

Note that if this is evaluated in shallow water, then $F_0 = a_0^2 h_0^{1/2}$, where a_0 is the wave amplitude at a location offshore where $h = h_0$. On the other hand, in deep water as $kh \to \infty, c_g \to g/2\omega$, and then $F_0 = a_\infty^2 g/2\omega$, where a_∞ is the constant wave amplitude in deep water. The surf zone $x < x_b, h < h_b = h(x_b)$ can then be defined by the criterion that h_b is that depth where $a/h = A_{cr}$, defining an empirical breaking condition. A suitable value is $A_{cr} = 0.44$, see Mei (1983) or Svendsen (2006).

The last step is to find the wave set-up $\bar{\zeta}$ from the mean momentum equation (12), which here becomes

$$S_x + gH\bar{\zeta}_x = 0\,, \tag{16}$$

$$\text{where} \quad S = \frac{c_g}{c} E \cos^2\phi + (\frac{c_g}{c} - \frac{1}{2})E\,. \tag{17}$$

Here ϕ is the angle between the wave direction and the onshore direction, and S is the "xx" component of the tensor $\mathbf{S}$. As $h \to 0$, $\phi \approx 0, c_g \approx c, S \approx 3E/2$, and we recover the well-known result obtained by Longuet-Higgins and Stewart (1962) of a wave set-down in the shoaling zone

$$\bar{\zeta} = -\frac{a^2}{4h} = -\frac{F_0}{4h^{3/2}}\,. \tag{18}$$

Here we have assumed that $\bar{\zeta}$ is zero far offshore.

2.4 Surf zone

In the surf zone $0 < x < x_b, 0 < h < h_b$, we make the usual assumption (see Mei (1983) for instance) that the breaking wave height $2a$ is proportional to the total depth H, so that

$$2a = \gamma\, H\,, \quad \text{or} \quad E = \frac{g\gamma^2 H^2}{8}\,, \tag{19}$$

Here the constant γ is determined empirically, and a typical value is $\gamma = 0.88$.. To determine the mean height $H = h + \bar{\zeta}$, we again use the mean momentum equation (12), but now assume that $S = 3E/2 = \Gamma g H^2/2$ where $\Gamma = 3\gamma^2/8$, so that

$$\Gamma H H_x + H(H-h)_x = 0\,, \quad \text{so that} \quad H = H_b + \frac{h - h_b}{(1+\Gamma)}\,, \tag{20}$$

where the constant $H_b = h_b + \bar{\zeta}_b$ is determined by requiring continuity of the total mean height at $x = x_b$. Note that using (18)

$$H_b = h_b - \frac{F_0}{4h_b^{3/2}}\,,$$

and since H_b must be positive, there is a restriction on either the offshore wave amplitude a_0 or a_∞ through F_0, or on the breaker depth h_b,

$$F_0 < 4h_b^{5/2} \,. \tag{21}$$

Note that the expression (20) is valid for any depth $h(x)$, although in the literature it is often derived only for a linear depth profile $h = \alpha x$.

We are now in a position to determine the displaced shoreline $x = x_s$, defined by the condition that $H = 0$. That is, if $h_s = h(x_s)$ then $H = (h - h_s)/1 + \Gamma$, where

$$h_s = -\Gamma h_b - (1 + \Gamma)\bar{\zeta}_b \,, \tag{22}$$

Note that to use the expression (22) it may be necessary to extend the definition $h(x)$ into $x < 0$. For instance for a linear beach, $h = \alpha x$, this is straightforward, but for a quadratic beach profile, $h = \beta x^2$, the extension for negative x should be $h = -\beta x^2$ say. Note that from (18),

$$\bar{\zeta}_b = -\frac{F_0}{4h_b^{3/2}} \,,$$

and, on combining this with the condition (21), it follows that the shoreline advances (recedes), that is $x_s < 0, h_s < 0$ ($x_s > 0, h_s > 0$) when

$$\begin{aligned} F_0 &< \frac{\Gamma}{1+\Gamma} 4h_b^{5/2} \,, \\ \text{or} \quad \frac{\Gamma}{1+\Gamma} 4h_b^{5/2} &< F_0 < 4h_b^{5/2} \,. \end{aligned} \tag{23}$$

Curiously, this anomalous result does not seem to have been noticed previously. Since there is an expectation that the shoreline should advance (see Dean and Dalrymple (2002) for instance), essentially it states that the present model is only valid for sufficiently small waves far offshore, defined by the first inequality in (23), which slightly refines the constraint (21).

3 A model of bottom sediment transport

3.1 Formulation of a sediment flux law

To take account of bottom sediment transport, we must allow $h = h(\mathbf{x}.t)$ to be a time-dependent dynamic variable. Hence the kinematic bottom boundary condition becomes

$$h_t + \mathbf{u}.\nabla h = -w\,, \quad \text{at} \quad z = -h(\mathbf{x}, t)\,, \tag{24}$$

Since h can now evolve with time, a second bottom boundary condition is needed which is an appropriate sediment flux law

$$h_t = \nabla.\mathbf{Q}\,, \tag{25}$$

where $\mathbf{Q}$ is the sediment flux, evaluated at the bottom. This equation is now averaged so that the sediment flux equation (25) becomes

$$\bar{h}_t \;=\; \nabla.\bar{\mathbf{Q}}\,, \tag{26}$$

where $\bar{\mathbf{Q}}$ is the wave-averaged sediment flux. In the literature various flux laws have been proposed, depending on the assumed sediment type. Here we follow the type of formulation used by Caballeria *et al* (2002), Calvete *et al* (2001, 2002), Garnier *et al* (2006), Restrepo (2001), and Lane and Restrepo (2007). From the latter in particular, but with some simplifications, we let

$$\bar{\mathbf{Q}} = \frac{\mu}{1 - p_s}(\mathbf{Q}_b + \mathbf{Q}_s)\,, \tag{27}$$

where $p_s \approx 0.4$ is the bed porosity, and $\mu \approx 0.05$ is a measure of how often the waves are large enough to move the sediment. The quantities $\mathbf{Q}_{b,s}$ are the bed-load and suspended sediment fluxes respectively, and are given by expressions of the form

$$\begin{aligned} \mathbf{Q}_b &= \nu_b(|\mathbf{u}_w|^2\mathbf{U} + \lambda_b|\mathbf{u}_w|^3\nabla b)\,, \\ \mathbf{Q}_s &= \nu_s(H|\mathbf{u}_w|^3\mathbf{U} + \lambda_s|\mathbf{u}_w|^5\nabla b)\,. \end{aligned} \tag{28}$$

Here $|\mathbf{u}_w|$ is the wave velocity magnitude, and we recall from (11) that $\mathbf{U} = \bar{\mathbf{u}} + \mathbf{U}_S$ is the Lagrangian mean velocity, where $\mathbf{U}_s$ is the Stokes drift velocity. $b = \bar{h} - h_0(x)$ is the deviation of $\bar{h}$ from a reference depth $h_0(x)$, which can be taken to be either the initial depth, or an equilibrium depth profile. The coefficients ν_b and ν_s are for bed-load and suspended transport respectively. These expressions are usually used outside the surf zone, and need modification inside the surf zone, where there are many various approaches, see the reviews by Roelvink and Broker (1993) and that in the Coastal Engineering Manual (2002). Here we assume that they remain valid, but only in qualitative form, inside the surf zone.

It can now be easily verified that the basic equation set, namely the equation for the conservation of waves (6) combined with the dispersion relation (4), the wave action equation (7), the averaged equation for conservation of mass (8) and the averaged equation for conservation of horizontal momentum (12), where the radiation stress tensor is given by (14), all continue to hold, with h replaced by $\bar{h}$. However, we must note that now $H = \bar{h} + \bar{\zeta}$. With this extended formulation we will re-evaluate the previously obtained expressions for the wave set-up. As in section 2, we assume there is no dependence in the transverse direction, and that $\mathbf{U} = (U, 0), \mathbf{U}_s = (U_s, 0)$.

3.2 Shoaling zone: effect of sediment transport

In the shoaling zone $\mathbf{u}_w \approx \tilde{\mathbf{u}}$ where $\tilde{\mathbf{u}}$ is given by the sinusoidal expression (5). Then, assuming that the waves are in shallow water it follows that $|\mathbf{u}_w|^2$ can be estimated as $ga^2/h = 2E/h$. Since in the present theory, the beach slope is assumed to be small, the diffusion terms in (28) will be omitted (but see section 3.4). It follows that the sediment flux (27) is expressed as

$$\bar{\mathbf{Q}}_{shoal} = \frac{2\mu E}{(1 - p_s)h}(\nu_b + \nu_s (2Eh)^{1/2})\mathbf{U}\,. \tag{29}$$

The Stokes drift is given by $\mathbf{U}_s = E\mathbf{k}/\omega h$, and is $O(E)$. Since we expect that the Eulerian mean flow $\bar{\mathbf{u}}$ forced by the waves will also be at least $O(E)$, it follows that $\mathbf{U}$ is $O(E)$. Hence, in the shoaling zone $\bar{\mathbf{Q}}_{shoal}$ is $O(E^2)$, and hence should be consistently neglected when compared with other wave-induced mean flow quantities. Hence in this preliminary study we shall ignore the effect of the mean sediment flux in $x > x_b$, and so $\bar{h} = h(x)$, $H = h(x) + \bar{\zeta}$ and there is no alteration to the equation system derived in section 2.2. The steady wave set-up solution can now be derived in the same manner described in section 2.3.

However, it transpires that now, in the surf zone $U \neq 0$, and in order to match at $x = x_b$, we may need to allow for $U \neq 0$ in the shoaling zone as well. Then, the conservation of mass equation implies that $HU = M = H_b U_b$, where as usual the subscript denotes the values at $x = x_b$. Then, instead of (16) we get that

$$-\frac{M^2 H_x}{H^2} + S_x + gH\bar{\zeta}_x = 0\,, \tag{30}$$

where S is again given by (17), and becomes $S \approx 3E/2$ as $h \to 0$. In place of (18) we now find that

$$\bar{\zeta} = -\frac{a^2}{4h} - \frac{M^2}{2gh^2}\,. \tag{31}$$

Note that the extra term always enhances the set-down. We show below that when $\nu << 1$, M is order ν, and hence this term can usually be neglected.

3.3 Surf zone: effect of sediment transport

In the surf zone, we assume that the flow is dominated the sediment transport is dominated by the Stokes drift $\mathbf{U}_s = (U_s, 0)$ where we estimate that $U_S \sim -|\mathbf{u}_w|^2/\sqrt{gH}$. Hence in the expression (28) we replace $U = U_s + \epsilon\bar{u}$ with $U_s + \epsilon\bar{u}$. Here ϵ is a small dimensionless parameter, introduced to account for weak offshore flow, especially near the shoreline. Although ϵ is unknown, we shall assume that $0 < \epsilon << 1$, and it will then transpire that its actual value is immaterial for the present model. Note that $U_s + \epsilon\bar{u}$ is more conveniently expressed as $(1-\epsilon)U_s + \epsilon U$. For the wave field in the surf zone we use the scaling $|\mathbf{u}_w|/(gH)^{1/2} \sim 2a/H$, so that again using the empirical expression (19), $|\mathbf{u}_w| \sim \gamma(gH)^{1/2}$. Again omitting the diffusive terms in (28), and setting $\bar{Q} = (Q_{surf}, 0)$ we obtain the expression,

$$\begin{aligned} Q_{surf} &= \nu\{-F(H) + G(H)U\}\,, &(32)\\ F(H) &= (1-\epsilon)H(gH)^{1/2}(1+\sigma(gH)^{3/2})\,, &(33)\\ G(H) &= \frac{\epsilon}{\gamma^2}H(1+\sigma(gH)^{3/2}) &(34)\end{aligned}$$

$$\text{where} \qquad \nu = \frac{g\gamma^4\mu\nu_b}{1-p_s}\,, \quad \sigma = \frac{\gamma\nu_s}{g\nu_b}\,. \tag{35}$$

Using the estimates $\nu_b = 1.8\times10^{-4}\,s^2\,m^{-1}$, $\nu_s = 1.0\times10^{-3}\,s^3\,m^{-3}$ the dimensionless small parameter $\nu = 0.88\times10^{-4}$ and $\sigma = 0.05\,s^3\,m^{-3}$.

The basic set of equations is then (26) using the expression (32), and (8, 12),

$$\begin{aligned} \bar{h}_t + \nu\{F(H) - G(H)U\}_x &= 0\,, &(36)\\ H_t + (HU)_x &= 0\,, &(37)\\ U_t + UU_x + g(1+\Gamma)H_x - g\bar{h}_x &= 0\,. &(38)\end{aligned}$$

Note again that now $H = \bar{h} + \bar{\zeta}$. Since for the wave set-up problem, $U = 0$ when $\nu = 0$ (that is, there is no sediment transport), we anticipate that when $\nu << 1$, then U is $O(\nu)$. It transpires that for the wave set-up solution of interest this is indeed the case, and hence the term $G(H)U$ in (36) is $O(\epsilon\nu)$ and can be neglected. It is omitted henceforth. Before proceeding we return to the full expressions (28) and note that the criteria for the neglect of the diffusive terms are $\lambda_b\alpha << 1, \lambda_s\alpha < (H/g)^{1/2}$ where α is a measure of the beach slope $\bar{h}_x$. With $\lambda_b = 0.7$, $\lambda_s = 2.5\,s$, $H = 2\,m$ this implies that we require that $\alpha << 1.4, 0.2$ respectively. This system forms a nonlinear hyperbolic system for $[H, U, \bar{h}]$. The most important difference from the analysis of section 2 is that equations (36, 37) show that $\bar{h}_t, H_t \neq 0$, and it then follows that the mean offshore velocity $U \neq 0$ in general, albeit rather small when $\nu << 1$.

We can write the system (36, 37, 38) as a 3×3 nonlinear hyperbolic system of the form

$$\mathbf{v}_t + \mathbf{A}(\mathbf{v})\mathbf{v}_x = 0\,, \quad \mathbf{v}^t = [H, U, \bar{h}]\,. \tag{39}$$

The eigenvalues λ of $\mathbf{A}$ for either system are given by

$$\det[\mathbf{A}(\mathbf{v}) - \lambda\mathbf{I}] = 0\,, \tag{40}$$

which leads to the cubic equation

$$\lambda\{g[1+\Gamma]H - (U-\lambda)^2\} - \nu g H F'(H) = 0 \tag{41}$$

The system is hyperbolic if this expression has three real roots. For $\nu << 1$, the roots are

$$\lambda_1 \approx \frac{\nu g H F'(H)}{\{g(1+\Gamma)H - U^2\}} \quad \lambda_{2.3} \approx U \pm [g(1+\Gamma)H]^{1/2}\,. \tag{42}$$

All are real-valued, and so in this limit the system is hyperbolic. The first root is the one of main interest here, as it is arises directly from the sediment transport term.

Nonlinear hyperbolic systems support a family of simple wave solutions, of the form

$$\mathbf{v} = \mathbf{v}(\alpha)\,, \tag{43}$$

where $\alpha = \alpha(x,t)$ is an arbitrary new variable, and could be taken as any one of the set $H, U, \bar{h}$. Substitution into (39) shows that

$$\alpha_t + c(\alpha)\alpha_x = 0\,, \quad \text{where} \quad c = \lambda\,, \tag{44}$$

is one of the eigenvalues of $\mathbf{A}$, and $\mathbf{v}_\alpha$ is then a corresponding eigenvector given by

$$\begin{aligned} -\lambda\bar{h}_\alpha + \nu F'(H)H_\alpha &= 0\,, && (45)\\ (U-\lambda)H_\alpha + HU_\alpha &= 0\,, && (46)\\ (U-\lambda)U_\alpha + g(1+\Gamma)H_\alpha - g\bar{h}_\alpha &= 0\,. && (47) \end{aligned}$$

We choose $\lambda = \lambda_1$, the root corresponding to the sediment transport term, and given approximately by (42) when $\nu << 1$. We then readily find that

$$\begin{aligned} c &= \frac{\nu F'(H)}{1+\Gamma} + \cdots\,, && (48)\\ U &= \frac{\nu F(H)}{(1+\Gamma)H} + \cdots\,, && (49)\\ H &= \frac{(\bar{h}+C_0)}{(1+\Gamma)} + \cdots\,. && (50) \end{aligned}$$

Here C_0 is a constant of integration, determined from a boundary matching condition. Because the relation (50) is conserved through shocks, see (58) below, this can be applied at $x = x_b$ so that,

$$C_0 = \Gamma h_b + (1+\Gamma)\bar{\zeta}_b\,, \qquad (51)$$

$$\text{so that} \quad H = H_b + \frac{\bar{h} - h_b}{(1+\Gamma)} + \cdots\,, \qquad (52)$$

which is the same expression as (20) with h replaced by $\bar{h}$. Note that U is $O(\nu)$ with $U > 0$. Thus in this simple wave solution, the mean flow is weak and offshore.

A hyperbolic system can also support discontinuous, or shock, solutions. Assuming that across a discontinuity, the sediment flux, mass and momentum are conserved, the shock conditions can be derived in the usual way by integrating across the discontinuity. If the shock speed is V, then these are readily obtained from the set (36, 37, 38),

$$\begin{aligned} -V[\bar{h}] + \nu[F(H)] &= 0\,, && (53)\\ -V[H] + [HU] &= 0\,, && (54)\\ -V[U] + [\frac{U^2}{2} + g(1+\Gamma)H - g\bar{h}] &= 0\,. && (55) \end{aligned}$$

Here $[\cdot]$ denotes the jump across the discontinuity. When $\nu << 1$, we see that the shock speed V of interest is $O(\nu)$. Assuming that, as above, also U is $O(\nu)$, the shock relations are approximated by

$$
\begin{aligned}
-V[\bar{h}] + \nu[F(H)] &= 0\,, &(56)\\
-V[H] + [HU] &= 0\,, &(57)\\
(1+\Gamma)[H] - [\bar{h}] &= 0\,. &(58)
\end{aligned}
$$

Only equation (57) involves U, and equation (58) can be used to eliminate $\bar{h}$ so that (56, 57) reduce to

$$-V(1+\Gamma)[H] + \nu[F(H)] = 0 \qquad (59)$$

$$[(1+\Gamma)HU - \nu F(H)] = 0\,. \qquad (60)$$

Since $F(H)$ is an increasing function of H, the expression (59) shows that the shock speed V is positive. Also the expressions (60) shows that the simple wave relation (49) for U is conserved across the shock.

In general, the system (39) is to be solved with the boundary conditions at $x = x_b$ that $[H, U, \bar{h}] = [H_b, M/H_b, h_b]$, where $H_b = h_b + \bar{\zeta}_b$. When we assume that the solution in the shoaling zone $x > x_b$ is steady, given by (18), then these boundary data are all known constants. The initial condition is more problematic to specify. Here we shall assume that in the surf zone $[H, U, \bar{h}] = [H_0(x), 0, h(x)]$ at $t = 0$, where $H_0(x)$ is the solution (20) in the absence of any sediment transport. This choice corresponds to turning on the sediment transport at $t = 0$, and is clearly an over-simplification of reality, but we expect the solution to be indicative of more realistic initial conditions. In effect, we are assuming that the wave field is turned on at $t = 0$ and reaches the sediment-free solution of section 2.4 instantaneously. That is we are assuming that the time scale for the steady sediment-free solution to be reached is much shorter than that for the sediment transport terms to take effect. Note that there is a discontinuity in U at the point $x = x_b, t = 0$, which requires that a shock emanates from this point. In the limit of interest when $\nu << 1$, the shock speed $V = O(\nu) > 0$, and we infer that a thin layer develops near $x = x_b$ but lying in $x > x_b$. In order to use the simple wave solution discussed above when $\nu << 1$ to solve this problem, we see that in this solution $U \neq 0$, albeit of $O(\nu)$. we infer that there is a thin boundary layer near $t = 0$ within which there an adjustment for U from zero to the simple wave value.

The simple wave solution can now be found by the method of characteristics, that is

$$\frac{dx}{dt} = c(\bar{h}),\ \frac{d\bar{h}}{dt} = 0\,, \quad \text{for} \quad x < x_b\,, \tag{61}$$

$$\bar{h} = h_b \quad \text{at} \quad x = x_b\,, \quad \bar{h} = h(x) \quad \text{at} \quad t = 0\,, \tag{62}$$

$$\text{so that} \quad x - c(\bar{h})t = x_0\,, \bar{h} = h(x_0) \quad \text{for} \quad x_0 < x_b\,, \tag{63}$$

where x_0 is the initial value of x along each characteristic. The solution can be written in the form

$$\bar{h} = h(x - c(\bar{h})t)\,, \quad \text{for} \quad x - c(h_b)t < x_b\,. \tag{64}$$

Since $c(\bar{h})$ is an increasing function of h, it can be shown that in this simple wave solution, $\bar{h}, H$ decrease at each fixed x as t increases, and also decrease for each fixed t as x decreases, since we have assumed $h_x(x) > 0$. Thus, in this solution the beach is continually replenished. Because the characteristics go offshore, they intersect a shock $x = x_s(t))$ emanating from $x = x_b, t = 0$, where the jump conditions (59) is imposed. Thus we get that

$$-V(H_b - H(x_s,t)) + \nu(F(H_b) - F(H(x_s,t)) = 0\,, \quad V = \frac{dx_s}{dt}\,. \tag{65}$$

Here $H(x_s,t)$ is obtained from the simple wave solution, and because the shock is thin, we have approximated the values of $H(x \to x_s+, t)$ as this at $x = x_b$ for simplicity. The expression (65) is a differential equation that determines the shock. This completes the solution. Also the expression (60) for U can now be used to find the value of M, that is

$$M = \frac{\nu F(H_b)}{(1+\Gamma)}\,. \tag{66}$$

A typical timescale can be estimated as the time for a characteristic from $x = 0$ to reach $x = x_b$, that is $t_s = x_b/c(\bar{h} = 0)$, or

$$t_s = \frac{(1+\Gamma)x_b}{\nu F'(H_0(0))}\,, \quad H_0(0) = \frac{C_0}{1+\Gamma} = H_b - \frac{h_b}{(1+\Gamma)}\,. \tag{67}$$

For instance, with the parameter values already specified, and with $x_b = 20\,m, H_b = 2\,m, h_b = 1.8\,m$, we find that $t_s = 4800\,s$ which is a reasonable value. THis estimate is dominated by the suspended sediment term, that with coefficient σ. If this is removed the estimated timescale is 7 times larger.

For a linear beach slope $h = \alpha x$, the simple wave solution (64) reduces to

$$\bar{h} = \alpha(x - c(\bar{h})t)\,, \tag{68}$$

For $\nu << 1$ this reduces to the expression

$$(1+\Gamma)H = \alpha x + C_0 - \frac{2\nu\alpha F'(H)t}{1+\Gamma}\,, \tag{69}$$

$$\text{where}\quad F'(H) = (1-\epsilon)\{\frac{3(gH)^{1/2}}{2} + 3\sigma(gH)^2\}\,.$$

The first expression reduces to (20) when $\nu \to 0$, but for $\nu > 0$ it is a quartic polynomial equation for $Y = (gH)^{1/2}$, which can be written as

$$Y^2 = X - 2T(Y + 2\gamma\sigma Y^4)\,, \quad T = \frac{3(1-\epsilon)\nu\alpha g t}{2(1+\Gamma)^2}\,, \quad X = \frac{g(\alpha x + C_0)}{1+\Gamma}\,. \tag{70}$$

Explicit solutions are not readily available, but it is readily shown that there is only one positive root for Y, for each given $X > 0, T > 0$. Also, the shoreline, where $Y = 0$ does note change in time, and remains at $X = 0$. However, explicit solutions can be found in the two limits when either bed-load or suspended sediment dominate. These correspond formally to the respective limits $\sigma \to 0, \sigma \to \infty$, when we get that

$$Y = \frac{X}{T + (T^2 + X)^{1/2}}\,, \quad \text{or} \quad Y^2 = \frac{2X}{1 + (1 + 16\gamma\sigma XT)^{1/2}}\,, \tag{71}$$

respectively. These are plotted in figure 1 for $\bar{h}$ as a function of h for a fixed value of $t = 1000, 6000$ when $\sigma = 0, \infty$ respectively.

3.4 Steady state

It is clear from (36, 37) that the present sediment transport model cannot allow any steady state to form, as $\bar{h}_t = H_t = 0$ would then imply that $H = 0$, which is unacceptable. Hence, if a steady state is to be reached, we must replace the sediment law (32) by an expression which takes account of the

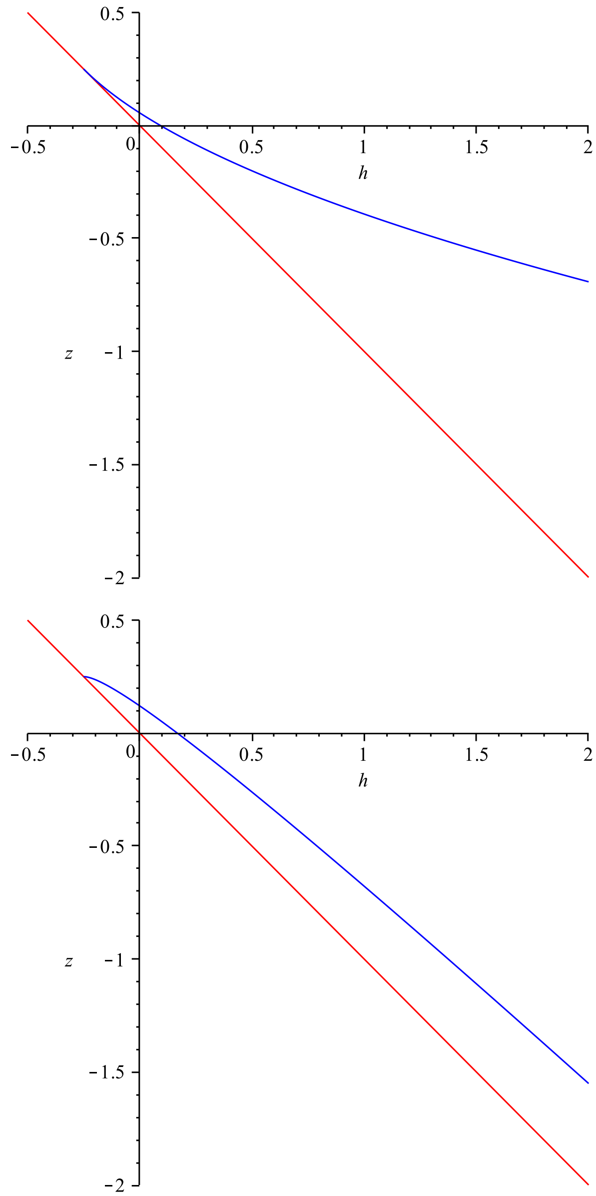

Figure 1: Plot of the solution (71) for $\bar{h}$ (blue) for the parameter setting $\gamma = 0.88, \alpha = 0.1, h_b = 2\,m, H_b = 1.73\,m, \epsilon = 0$ for $t = 1000\,s$ for the case $\sigma = 0$ in the upper panel, and at $t = 6000\,s$ for the case $\sigma \to \infty$ in the lower panel. In each case the red line is h, the value at $t = 0$.

beach slope through the diffusive terms in (28). Thus, from the discussion in section 3.3 we now replace (32) with

$$\begin{aligned} Q_{surf} &\approx \nu\{-F(H) + D(H)b_x) \,, \\ D(H) &= \frac{\lambda_b}{\gamma} H(gH)^{1/2} + \sigma\lambda_s\gamma^2 gH^2(gH)^{1/2} \,. \end{aligned} \tag{72}$$

Consequently equation (36) is replaced by

$$\bar{h}_t = \nu\{-F(H) + D(H)b_x\}_x \,, \quad b = \bar{h} - h_0(x) \,. \tag{73}$$

The remaining two equations (37, 38) are unchanged. Equation (73) has the structure of a nonlinear diffusion equation, and so there is a possibility that a steady-state can be achieved. Indeed if we assume that there is a steady-state solution then $U = 0$, (73) implies that $\bar{Q}_{surf} = 0$, and then

$$F(H) - D(H)b_x = 0 \,. \tag{74}$$

Equation (38) can be integrated to yield

$$H(1 + \Gamma) = \bar{h} + \text{constant} \,. \tag{75}$$

Substituting (75) into (74) yields

$$1 + \gamma\sigma(gH)^{3/2} = (\lambda_b + \sigma\lambda_s\gamma gH)((1 + \Gamma)H_x - h_{0x}) \,. \tag{76}$$

Clearly the solution will depend on the choice of the reference depth, and here we make the simple choice that $h_0(x) = \text{constant} + \alpha_0 x$. The general solution can now be found by quadrature. However it is more instructive to consider the two limits $\sigma \to 0, \infty$, which correspond to the cases when either bed-load or suspended transport dominates. These limits yield respectively

$$\sigma = 0 : \quad H = \frac{\tilde{\alpha}(x - x_s)}{(1 + \Gamma)} \,, \quad \tilde{\alpha} = \alpha_0 + \frac{\gamma}{\lambda_b} \,, \tag{77}$$

$$\sigma \to \infty : \quad (gH)^{1/2} - C_0 \log{(1 + \frac{(gH)^{1/2}}{C_0})} = \frac{\gamma(x - x_s)}{\lambda_s(1 + \Gamma)} \,, \tag{78}$$

$$\text{where} \qquad C_0 = \frac{\alpha_0 g \lambda_s}{\gamma} \,.$$

Here $x = x_s$ is the shoreline, and the corresponding expressions for $\bar{h}$ are recovered from (75). In the case $\sigma = 0$, the profile is just a linear slope but enhanced over the reference slope α_0. In the case $\sigma \to \infty$, we note that when $\alpha_0 = 0, C_0 = 0$ and (77) reduces to a quadratic expression in $H = C_1(x - x_s)^2, C_1 = \gamma^2/g\lambda_s^2(1 + \Gamma)^2$, while when $C_0 \to \infty$, the profile is again a quadratic expression, but now $4H \to C_1(x - x_s)^2$. Also, for this same case as $H \to 0$, again $4H \to C_1(x - x_s)^2$, while as $H \to \infty$, $H \to C_1(x - x_s)^2$. In effect the entire solution is close to some parabolic profile. For intermediate values of σ the solution varies between the linear slope (77) and the expression defined by (78). We infer that these equilibrium beach profiles range between a linear and a parabolic profile, and can probably be well approximated by a power law $(x - x_s)^\beta, 1 \leq \beta \leq 2$. However, it seems that the well-known Dean's law (Dean 1991, Dean and Darlymple 2002) when the profile is proportional to $(x - x_s)^{2/3}$ is not described by the present class of solutions. Indeed, to obtain Dean's law by the present approach requires that $D(H)/F(H) \propto (gH)^{1/2}$, Examining the formula (28), we see that this would require a stronger dependence on $|\mathbf{u}_w|$ in the diffusive term than this formula allows for. Whether or not the unsteady simple wave solutions of the previous subsections will eventually reach a steady state requires numerical solutions of (37, 38, 73), and will not be investigated here.

4 Summary and discussion

In this paper we have augmented the usual wave-averaged mean field equations, described in section 2, commonly used to describe wave set-up and wave-induced mean currents in the near-shore zone, with a commonly accepted sediment flux law (28). In this model, any sediment movement in the shoaling zone is ignored as being $O(E^2)$, and instead our focus is on how the augmented model modifies wave set-up in the surf zone. Here the sediment flux law is modelled empirically, based on (28), but with a modification to reflect the dominant effect of the Stokes drift term, leading to (32). Our main result in section 3.3 is that, when the diffusive terms in the flux law are ignored, then there is no steady-state set-up, and instead the mean bottom depth $\bar{h}$ in the surf zone evolves according to a simple wave equation. This is solved to yield a prediction that the beach is replenished. In section 3.4 we show that if the diffusive terms in the sediment flux law (32) are retained, then the simple wave equation, whose solutions are intrinsically unsteady, is replaced by a nonlinear diffusion equation (73) which allows a steady-state solution. This can be well represented by a power-law profile with index varying between one and two, that is between linear and parabolic profiles.

Although our present model makes a specific choice of the empirical parameters in (28), we would expect that other choices will lead to qualitatively similar results to those obtained here. A more serious limitation of the present model is that the outer boundary of the surf zone $x = x_b$ is assumed here to be fixed for all time. When sediment transport is allowed, the wave set-up becomes unsteady, and our solution indeed indicates that x_b will also be unsteady, and migrate offshore as the mean total depth decreases in the surf zone. This issue will await future study. Also, the present model is entirely one-dimensional, and it would be interesting to examine the stability of the solutions found here to transverse perturbations,

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